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Differential Geometry Lecture Notes Entitled :

Lectures and Solved Exercises in Differential Geometry

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Intended for : Fourth Year Students in the Secondary Mathematics Teacher Training Program.

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Introduction

These course notes, entitled Lectures and Solved Exercises in Differential Geometry, have been carefully prepared as part of the official content for the M413 Differential Geometry Module at the Ecole Normale Supérieure de Bou Saâda. They are intended primarily for Fourth Year Students in the Secondary Mathematics Teacher Training Program, but are equally suitable for any learner seeking a rigorous and accessible introduction to the subject.

Our objective in designing this material is to present the foundational ideas of differential geometry in a manner that is logically structured, pedagogically sound, and mathematically rigorous. The notes aim to cultivate both theoretical understanding and computational skill, with a particular focus on concepts and methods that are essential in both pure mathematics and applied contexts, such as physics and engineering.

The course is organized into three main chapters:

Chapter 1: A Review of Differential Calculus in $\mathbb{R}^n$

This chapter establishes the analytic foundations, covering differentiability, gradients, Jacobian matrices, directional derivatives, and the implicit function theorem. Numerous examples and counterexamples are included to highlight subtleties and guide intuition.

Chapter 2: Submanifolds of $\mathbb{R}^n$ and Abstract Manifolds

This chapter explore the concept of a submanifold from various perspectives—through local charts, submersions, immersions, and parametrizations—alongside topics such as tangent spaces and differentiability on submanifolds. The chapter concludes with a brief introduction

to abstract manifolds.

Chapter 3: Differential Forms and Integration

This chapter introduces the algebra of differential forms and their key operations, including the wedge product, exterior derivative, and pullback. These tools are then used to formalize line and surface integrals, culminating in the unified treatment of classical theorems: Green's, Stokes', and Gauss-Ostrogradsky's.

Each chapter is designed to be self-contained, enriched with detailed examples, explanatory remarks, and fully worked solutions to a broad range of exercises. Particular care has been taken to highlight common misunderstandings, reinforce key ideas through repetition, and provide geometric interpretations wherever appropriate.

It is our hope that these notes will serve not only as a comprehensive guide for students during the course, but also as a lasting reference for future teaching, research, or further study in geometry, topology, and mathematical analysis.

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Chapter 1

A Review of Differential Calculus in $\mathbb{R}^n$

In this chapter, the analytic foundations are established: differentiability, partial derivatives, gradient, Jacobian, immersions, submersions, diffeomorphisms, and the Implicit Function Theorem, with worked exercises. Draws on Aime [1], Hélein-Chaudouard [5], Renard [6], Tu [7], and Waldspurger [9].

1.1 Differentiability at a Point

Throughout this chapter, U will denote a non-empty open subset of $\mathbb{R}^n$, unless otherwise specified.

Definition 1.1. Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function, and let $a \in U$. We say that f is differentiable at a if there exists a continuous linear map $L_a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that:

$$f(a+h) = f(a) + L_a(h) + \|h\| \varepsilon(h) \quad \text{with } \lim_{h \longrightarrow 0} \|\varepsilon(h)\| = 0.$$

The linear map L_a is called the differential of f at a and is denoted by df_a.

Proposition 1.1. The linear map L_a introduced in Definition 1.1, when it exists, is unique.

Proof. Let $(e_i)_{1\leq i\leq n}$ be the canonical basis of $\mathbb{R}^n$. Suppose there exist two linear maps L_a^1 and L_a^2 satisfying the conditions of Definition 1.1. Then we have

$$f(a+h) = f(a) + L_a^1(h) + \|h\|\varepsilon^1(h) \quad \text{with} \lim_{h\longrightarrow 0} \left\|\varepsilon^1(h)\right\| = 0,$$

and

$$f(a+h) = f(a) + L_a^2(h) + \|h\|\varepsilon^2(h) \quad \text{with} \lim_{h\longrightarrow 0} \left\|\varepsilon^2(h)\right\| = 0.$$

Subtracting the two expressions yields

$$L_a(h) = -\|h\|\varepsilon(h) \quad \text{with} \lim_{h\longrightarrow 0} \|\varepsilon(h)\| = 0,$$

where $L_a = L_a^1 - L_a^2$ and $\varepsilon = \varepsilon^1 - \varepsilon^2$. For $h = te_i$ with $t > 0$, we get

$$L_a(te_i) = -\|te_i\|\varepsilon(te_i) \quad \text{with} \lim_{t\longrightarrow 0} \|\varepsilon(te_i)\| = 0.$$

Using the linearity of L_a and simplifying, we find:

$$L_a(e_i) = -\|e_i\|\varepsilon(te_i) \quad \text{with} \lim_{t\longrightarrow 0} \|\varepsilon(te_i)\| = 0.$$

Thus, $L_a(e_i) = 0$, for all $i = 1, ..., n$. Consequently $L_a = 0$, which implies $L_a^1 = L_a^2$. Therefore, the linear map L_a is unique.

Definition 1.2. Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable function on U. The differential function of f, denoted df, is defined as follows:

$$df : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m), \ x \longmapsto df_x,$$

where $L(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

For each $x \in U$, the linear map df_x is represented (with respect to the canonical basis) by the Jacobian matrix of f at x. It provides the best linear approximation of$f$ near x, describing the infinitesimal variation of f in different directions at that point.

Example 1.1. Let f be a linear form on $\mathbb{R}^n$. We have

$$\forall x, h \in \mathbb{R}^n : f(x+h) = f(x) + f(h),$$

which gives

$$\frac{f(x+h)-f(x)-f(h)}{\|h\|}=0.$$

It follows directly from Definition 1.1 that f is differentiable on $\mathbb{R}^n$, and for all $x\in\mathbb{R}^n$, we have $df_x=f$. The differential function of f is the constant map defined by

$$df:\mathbb{R}^n\longrightarrow L(\mathbb{R}^n,\mathbb{R}),\ x\longmapsto df_x=f.$$

Example 1.2. Let $I\subset\mathbb{R}$ be an open interval, and let $f:I\longrightarrow\mathbb{R}$ be a function. We will show that the derivability of f on I implies its differentiability, and vice versa. Suppose first that f is derivable on I, and let $x\in I$. Then we have

$$\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)}{h}=f'(x),$$

which gives

$$\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)-f'(x)h}{|h|}=0.$$

Since the map

$$L_x:\mathbb{R}\longrightarrow\mathbb{R},\ h\longmapsto f'(x)h,$$

is linear, it follows that f is differentiable at x. For all $x\in I$ and $h\in\mathbb{R}$, we have: $df_x(h)=f'(x)h$. Since df_x is a linear map, we get $df_x(h)=hdf_x(1)=f'(x)h$ for all $h\in\mathbb{R}$, which implies $df_x(1)=f'(x)$ for all $x\in I$.

Conversely, suppose that f is differentiable on I, i.e.,

$$\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)-df_x(h)}{|h|}=0,$$

which in turn implies

$$\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)-df_x(h)}{h}=0.$$

Since $df_x(h)=hdf_x(1)$, we get

$$\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)}{h}=df_x(1),$$

which proves that f is derivable at x and that $f'(x)=df_x(1)$.

1.2 Partial Derivatives, Gradient, and Jacobian Matrix

Definition 1.3. Let $a \in U$ and let $\nu \in \mathbb{R}^n$ such that $\|\nu\| = 1$. We define $D = \{t \in \mathbb{R} : a + t\nu \in U\}$, and consider the function $\psi : D \longrightarrow \mathbb{R}$ defined by: $\psi(t) = f(a + t\nu)$. If ψ is differentiable at the point 0, we denote its derivative by $\psi'(0)$, which is called the directional derivative of f in the direction of ν. We write:

$$\frac{\partial}{\partial \nu} f(a) = \lim_{t \longrightarrow 0} \frac{f(a + t\nu) - f(a)}{t}$$

In particular, for the canonical basis vectors e_i, we have:

$$\frac{\partial}{\partial e_i} f(a) = \frac{\partial f}{\partial x_i}(a), \ i = 1, ..., n.$$

Proposition 1.2. If $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at a point $a \in U$. Then all partial derivatives of f exist at a. In fact, one has

$$\frac{\partial f}{\partial x_i}(a) = df_a(e_i), \ i = 1, ..., n,$$

and df_a is given by

$$df_a(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) h_i, \quad \forall h = (h_1, ..., h_n) \in \mathbb{R}^n,$$

where $\{e_1, ..., e_n\}$ denotes the canonical basis of $\mathbb{R}^n$.

Proof. The differentiability hypothesis of f at a is written as

$$f(a + h) = f(a) + df_a(h) + \|h\| \varepsilon(h) \quad \text{with } \lim_{h \longrightarrow 0} \|\varepsilon(h)\| = 0.$$

For a fixed i and $h = h_i e_i$, the previous relation gives

$$f(a + h_i e_i) = f(a) + df_a(h_i e_i) + \|h_i e_i\| \varepsilon(h_i e_i) \quad \text{with } \lim_{h_i \longrightarrow 0} \|\varepsilon(h_i e_i)\| = 0,$$

or, taking into account that df_a is a linear map, we can rewrite

$$\frac{f(a + h_i e_i) - f(a)}{h_i} = df_a(e_i) + \frac{|h_i|}{h_i} \|e_i\| \varepsilon(h_i e_i) \ \text{ with } \lim_{h_i \longrightarrow 0} \|\varepsilon(h_i e_i)\| = 0.$$

This simplifies to

$$\lim_{h_i \longrightarrow 0} \frac{f(a + h_i e_i) - f(a)}{h_i} = df_a(e_i).$$

Thus, f has a partial derivative with respect to x_i at a, and we have

$$\frac{\partial f}{\partial x_i}(a) = df_a(e_i), \ \forall i = 1, ..., n,$$

which gives

$$df_a(h) = df_a\left(\sum_{i=1}^{n} h_i e_i\right) = \sum_{i=1}^{n} h_i df_a(e_i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) h_i.$$

The converse of Proposition 1.2 is not true: the existence of all directional derivatives at a point does not guarantee differentiability at that point. To make this distinction clear, we present the following example:

Example 1.3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}.$$

For $h = (h_1, h_2) \neq (0, 0)$, we have

$$\lim_{t \longrightarrow 0} \frac{f(th_1, th_2) - f(0, 0)}{t} = \lim_{t \longrightarrow 0} \frac{t^3 h_1^2 h_2}{t^3 (h_1^2 + h_2^2)} = \frac{h_1^2 h_2}{h_1^2 + h_2^2}.$$

Thus, the function f has a derivative in every direction h at the point $(0, 0)$, given by

$$\frac{\partial}{\partial h} f(0, 0) = \frac{h_1^2 h_2}{h_1^2 + h_2^2}.$$

Now, let us show that f is not differentiable at $(0, 0)$. A simple calculation shows that

$$\frac{\partial}{\partial x} f(0, 0) = \frac{\partial}{\partial y} f(0, 0) = 0.$$

If f were differentiable at $(0, 0)$, then we would have

$$\lim_{(x,y) \longrightarrow (0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = 0.$$

We now show that this limit does not exist. For $y = \lambda |x|$ with $\lambda \neq 0$, we have

$$\frac{f(x, \lambda |x|)}{\sqrt{x^2 + \lambda^2 |x|^2}} = \frac{\lambda x^2 |x|}{\left(x^2 + \lambda^2 |x|^2\right)^{3/2}} = \frac{\lambda}{(1 + \lambda^2)^{3/2}}.$$

Thus, the limit of $\frac{f(x,y)}{\sqrt{x^2+y^2}}$ does not exist as (x,y) goes to $(0,0)$. Therefore, the function f is not differentiable at $(0,0)$.

Definition 1.4. Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that all first-order partial derivatives of f exist at a point $a \in U$. The gradient of f at a, denoted grad $f(a)$ or $\nabla f(a)$, is the vector in $\mathbb{R}^n$ defined by

$$\operatorname{grad} f(a) = \left(\frac{\partial f}{\partial x_i}(a)\right)_{1\leq i\leq n} = \left(\frac{\partial f}{\partial x_1}(a), ..., \frac{\partial f}{\partial x_n}(a)\right).$$

Definition 1.5. Let $F = (F_1, ..., F_m) : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector-valued function such that all first-order partial derivatives $\frac{\partial F_i}{\partial x_j}$ exist at a point $a \in U$. The Jacobian matrix of F at a, denoted by $J_F(a)$, is the matrix with m rows and n columns, defined by

$$J_F(a) = \left(\frac{\partial F_i}{\partial x_j}(a)\right)_{1\leq i\leq m,\ 1\leq j\leq n} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(a) & ... & \frac{\partial F_1}{\partial x_j}(a) & ... & \frac{\partial F_1}{\partial x_n}(a) \\ ... & ... & ... & ... & ... \\ \frac{\partial F_i}{\partial x_1}(a) & ... & \frac{\partial F_i}{\partial x_j}(a) & ... & \frac{\partial F_i}{\partial x_n}(a) \\ ... & ... & ... & ... & ... \\ \frac{\partial F_m}{\partial x_1}(a) & ... & \frac{\partial F_m}{\partial x_j}(a) & ... & \frac{\partial F_m}{\partial x_n}(a) \end{bmatrix}.$$

This matrix is also denoted by $\frac{\partial(F_1,...,F_m)}{\partial(x_1,...,x_n)}(a)$ or $\frac{D(F_1,...,F_m)}{D(x_1,...,x_n)}(a)$.

Remark 1.1. Using the concepts of gradient and Jacobian, differentials can be expressed in the following forms:

$$df_a(h) = \langle \operatorname{grad} f(a), h\rangle \quad \text{and } df_a(h) = J_F(a)\,h, \quad \forall h = (h_1, ..., h_n) \in \mathbb{R}^n,$$

where the symbol $\langle\ ,\ \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$, defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \forall x = (x_1, ..., x_n),\ y = (y_1, ..., y_n) \in \mathbb{R}^n.$$

1.3 Immersions, Submersions, and Diffeomorphisms

Definition 1.6. Let $f : U \longrightarrow \mathbb{R}$ be a function. We say that f is of class C^k on U , with $k \in \mathbb{N}^*$, if all partial derivatives of f up to order k exist and are continuous on U.

If $f \in C^k(U)$ for all $k \in \mathbb{N}^*$, we say that f is of class C^∞ on U, and we write $f \in C^\infty(U)$.

Example 1.4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} x^2 y \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

It is easy to verify that $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ exist on $\mathbb{R}^2$ and are given by

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y\left(2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases},$$

and

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

Let us show that $\frac{\partial f}{\partial x}$ is not continuous at the points $(0, y_0)$ where $y_0 \in \mathbb{R}^*$. To this end, we consider the sequence of points $\left(\frac{1}{2n\pi}, y_0\right) \in \mathbb{R}^2$. This sequence goes to $(0, y_0)$ when n goes to ∞; however,

$$\lim_n \frac{\partial f}{\partial x}\left(\frac{1}{2n\pi}, y_0\right) = -y_0 \neq \frac{\partial f}{\partial x}(0, y_0) = 0.$$

Thus, f is not of class C^1 on $\mathbb{R}^2$.

Proposition 1.3. Let $f : U \longrightarrow \mathbb{R}$ be a function. If f is of class C^1 on U, then f is differentiable on U.

Remark 1.2. The converse of Proposition 1.3 is not true; that is, there exist functions that are differentiable but not of class C^1. In other words, the set of C^1 functions on U is strictly contained within the set of differentiable functions.

Example 1.5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} x^2 y \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

We saw in Example 1.4 that f is not of class C^1 on the set $E = \{(0,y) \in \mathbb{R}^2,\ y \in \mathbb{R}^*\}$. We

now need to prove the differentiability of f on E. We have

$$\frac{\left|f(h,y+k)-f(0,y)-\left(h\frac{\partial f}{\partial x}(0,y)+k\frac{\partial f}{\partial y}(0,y)\right)\right|}{|h|+|k|}=\begin{cases}\frac{\left|h^2(y+k)\sin\left(\frac{1}{h}\right)\right|}{|h|+|k|} & \text{if } h\neq 0,\\ 0 & \text{if } h=0.\end{cases}$$

$$\leq |h((y+k))|.$$

This implies that

$$\lim_{(h,k)\to(0,0)}\frac{\left|f(h,y+k)-f(0,y)-\left(h\frac{\partial f}{\partial x}(0,y)+k\frac{\partial f}{\partial y}(0,y)\right)\right|}{|h|+|k|}=0.$$

Thus, the function f is differentiable at every point in E and $df_{(0,y)}=0$ for all $y\in\mathbb{R}^*$.

Definition 1.7. A function $f:U\subset\mathbb{R}^n\longrightarrow V\subset\mathbb{R}^m$ is called a diffeomorphism if:

1. f is bijective.
2. f is of class C^1 on U.
3. f^{-1} is of class C^1 on V.

Definition 1.8. A function $f:U\subset\mathbb{R}^n\longrightarrow V\subset\mathbb{R}^m$ is called a C^k-diffeomorphism ($k\geq 1$) if:

1. f is bijective.
2. f is of class C^k on U.
3. f^{-1} is of class C^k on V.

If f is a C^k-diffeomorphism for every $k\in\mathbb{N}^*$, then f is called a C^∞-diffeomorphism.

Remark 1.3. If $k=0$, the function f is called a homeomorphism, and it satisfies the following conditions:

1. f is bijective.

2. f is continuous on U.

3. f^{-1} is continuous on V.

Definition 1.9 (Immersion)**.** Let f be a function from U to $\mathbb{R}^m$. We say that f is an immersion at a point x in U if df_x is injective. If f is an immersion at every point of U, we say that f is an immersion on U.

Definition 1.10 (Submersion)**.** Let f be a function from U to $\mathbb{R}^m$. We say that f is a submersion at a point x in U if df_x is surjective. If f is a submersion at every point on U, we say that f is a submersion on U.

Remark 1.4. The function f can be an immersion only if $n \leq m$, and it can be a submersion only if $n \geq m$.

Theorem 1.1 (Normal Form of Immersions). Suppose that $0_{\mathbb{R}^n} \in U$ and $f(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$. If f is an immersion at $0_{\mathbb{R}^n}$, then there exists an open neighbourhood U' of $0_{\mathbb{R}^n}$ and a C^k-diffeomorphism ψ from a neighbourhood of $0_{\mathbb{R}^m}$ to a neighbourhood of $0_{\mathbb{R}^m}$ such that

$$\forall (x_1, ..., x_n) \in U',\ \psi \circ f(x_1, ..., x_n) = (x_1, ..., x_n, 0, ..., 0).$$

Theorem 1.2 (Normal Form of Submersions). Suppose that $0_{\mathbb{R}^n} \in U$ and $f(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$. If f is a submersion at $0_{\mathbb{R}^n}$, then there exist open neighbourhoods U_1, U_2 of $0_{\mathbb{R}^n}$ and a C^k-diffeomorphism $\phi : U_1 \longrightarrow U_2$ such that

$$\forall (x_1, ..., x_n) \in U_1,\ f \circ \phi(x_1, ..., x_n) = (x_1, ..., x_m).$$

1.4 Composition of Differentiable Functions

Let n, $m \in \mathbb{N}^*$. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. Consider functions $F : U \longrightarrow V$ and $G : V \longrightarrow \mathbb{R}$, and let $a \in U$.

Theorem 1.3. If F is differentiable at a and G is differentiable at $F(a)$, then the function $G \circ F$ is differentiable at a, and its differential is given by

$$d(G \circ F)_a = dG_{F(a)} \circ dF_a.$$

1.5 Implicit Function Theorems

A well-known result for functions of a single variable states that if f is of class C^1 on an interval I and $f'(x_0) \neq 0$ for $x_0 \in I$, then f is strictly monotonic on a neighbourhood $I_0 \subset I$ of x_0 and defines a bijection between the intervals I_0 and $f(I_0)$. Furthermore, the inverse bijection is also of class C^1. This result generalizes in higher dimensions to a theorem known as the Local Inversion Theorem.

Theorem 1.4. Let f be a function of class C^1 on an open set $U \subset \mathbb{R}^n$ with values in $\mathbb{R}^n$, and let $x \in U$ such that the Jacobian matrix $J_f(x)$ is invertible. Then, there exist two open sets, $U' \subset U$ and V such that $x \in U'$, $f(x) \in V$ and f is a C^1-diffeomorphism from U' to V.

Theorem 1.5. Let U be an open set in $\mathbb{R}^2$, and let (a, b) be a point in U. Let f be a function from U to $\mathbb{R}$ such that

1. f is of class C^1 on U.
2. $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$.

Then, there exist an open interval I containing a, an open interval J containing b and a unique C^1 function $\varphi : I \longrightarrow J$ such that:

$$I \times J \subset U, \quad \varphi(a) = b, \quad f(x, \varphi(x)) = 0 \ \ \forall x \in I \ \text{ and } \ \varphi'(x) = -\frac{\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))}.$$

Example 1.6. Consider the equation of a circle centered at the origin:

$$f(x, y) = x^2 + y^2 - r^2 = 0,$$

where $r > 0$ is a positive constant (the radius of the circle).

Let's examine whether $f(x, y)$ satisfies the conditions of the implicit function theorem near a particular point, say $(a, b) = (r, 0)$ and take U as an open set in $\mathbb{R}^2$ containing $(r, 0)$. Clearly, f is of class C^1 on U, and we have

$$\frac{\partial f}{\partial x}(r, 0) = 2r, \ \frac{\partial f}{\partial y}(r, 0) = 0.$$

Since $\frac{\partial f}{\partial x}(r, 0) \neq 0$ but $\frac{\partial f}{\partial y}(r, 0) = 0$ we can use the implicit function theorem to solve for x as a function of y near the point $(r, 0)$, but not y as a function of x. There exist an open interval I containing r, an open interval J containing 0 and a unique C^1 function $\varphi : I \longrightarrow J$ such that:

$$I \times J \subset U, \quad \varphi(r) = 0, \quad f(x, \varphi(x)) = 0 \ \ \forall x \in I \ \text{ and } \ \varphi'(0) = -\frac{\frac{\partial f}{\partial y}(r, 0)}{\frac{\partial f}{\partial x}(r, 0)} = 0.$$

Idea of proof of theorem 1.5. Since $\frac{\partial f}{\partial y}(a, b) \neq 0$, we can assume without loss of generality that $\frac{\partial f}{\partial y}(a, b) > 0$. Because f is of class C^1 on U, $\frac{\partial f}{\partial y}$ is continuous, so $\frac{\partial f}{\partial y}(x, y) > 0$ for all $(x, y) \in I \times J$. This implies that $f(a, y)$ is strictly increasing on J. Since $f(a, b) = 0$ and $f(a, y)$ is strictly increasing on J, then

$$\exists y_1 \in J : \ f(a, y_1) > 0$$

$$\exists y_2 \in J : \ f(a, y_1) < 0$$

Since f is of class C^1, it is continuous. Therefore, for x near a, we have $f(x, y_1) > 0$ and $f(x, y_1) < 0$. Since $\frac{\partial f}{\partial y}(x, y) > 0$ in $I \times J$, $f(x, y)$ is increasing as a function of y for each x near a (i.e. $x \in I$). Thus, by the intermediate value theorem, there exists a unique $y \in J$ such that $f(x, y) = 0$ for each x near a. This proves that there exists a unique function $y = \varphi(x)$ for all x in I such that $f(x, \varphi(x)) = 0$.

The implicit function theorem generalizes to functions of more than two variables.

Theorem 1.6. Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}$, and let $(a, b) \in U$ where $a = (a_1, ..., a_n) \in$

$\mathbb{R}^n$ and $b \in \mathbb{R}$. Consider the C^1 function on U

$$\begin{array}{rccl} f: & U \subset \mathbb{R}^n \times \mathbb{R} & \to & \mathbb{R} \\ & (x,y) & \mapsto & f(x,y). \end{array}$$

Assume that $f(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) \neq 0$. Then there exists an open neighbourhood A containing a, an open interval J containing b and a unique function $\varphi : A \longrightarrow J$ of class C^1 on A, such that $A \times J \subset U$ and for all $x \in A$, we have

$$f(x,\varphi(x)) = 0 \text{ and } \nabla\varphi(x) = -\frac{\left(\frac{\partial f}{\partial x_1}(x,\varphi(x)),...,\frac{\partial f}{\partial x_n}(x,\varphi(x))\right)}{\frac{\partial f}{\partial y}(x,\varphi(x))}$$

The previous result can be generalized to functions with more than two variables and vector-valued functions. The condition that arises here is the invertibility of a suitable partial derivative of the function.

Theorem 1.7. Let f be a C^k function defined on an open set $\Omega \subset \mathbb{R}^p \times \mathbb{R}^q$, with values in $\mathbb{R}^q$. Let $(a,b) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $f(a,b) = 0$. Assume that $d_y f(a,b) \in \text{Isom}(\mathbb{R}^q)$, i.e., the matrix

$$d_y f_{(a,b)} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(a,b) & ... & \frac{\partial f_1}{\partial y_q}(a,b) \\ ... & ... & ... \\ \frac{\partial f_q}{\partial y_1}(a,b) & ... & \frac{\partial f_q}{\partial y_q}(a,b) \end{pmatrix}$$

is invertible. Then, there exists an open neighbourhood U_a of a in $\mathbb{R}^p$, an open neighbourhood V_b of b in $\mathbb{R}^q$ and a C^k function $\varphi : U_a \longrightarrow V_b$ such that for all $(x,y) \in U_a \times V_b$, we have

$$f(x,y) = 0 \Longleftrightarrow y = \varphi(x).$$

The differential of φ at a is given by the following formula:

$$d\varphi_a = -\left[d_y f_{(a,b)}\right]^{-1} \circ d_x f_{(a,b)},$$

$$J_\varphi(a) = -\begin{pmatrix} \frac{\partial f_1}{\partial y_1}(a,b) & ... & \frac{\partial f_1}{\partial y_q}(a,b) \\ ... & ... & ... \\ \frac{\partial f_q}{\partial y_1}(a,b) & ... & \frac{\partial f_q}{\partial y_q}(a,b) \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a,b) & ... & \frac{\partial f_1}{\partial x_p}(a,b) \\ ... & ... & ... \\ \frac{\partial f_q}{\partial x_1}(a,b) & ... & \frac{\partial f_q}{\partial x_p}(a,b) \end{pmatrix}.$$

If f is of class C^k with $k \geq 1$, then φ is also of class C^k.

Remark 1.5. The implicit function theorem can be applied as soon as the matrix $d_y f_{(a,b)}$ is invertible. Then, according to the theorem, there exist open neighbourhoods U_a of a and V_b of b and a function $\varphi : U_a \longrightarrow V_b$, $\varphi(x) = (\varphi_1(x), ..., \varphi_q(x))$ such that:

1. $\varphi(a) = b$.

2. For all $(x, y) \in U_a \times V_b$, we have

$$\begin{cases} f_1(x_1, ..., x_p, y_1, ..., y_q) = 0 \\ \qquad ... \\ f_q(x_1, ..., x_p, y_1, ..., y_q) = 0 \end{cases} \iff \begin{cases} y_1 = \varphi_1(x_1, ..., x_p) \\ \qquad ... \\ y_q = \varphi_q(x_1, ..., x_p) \end{cases}.$$

Example 1.7. Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by: $f(x, y, z) = (x^2 - y^2, x^2 - z^2)$. Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that $f(x_0, y_0, z_0) = 0$. The Jacobian matrix of f at (x_0, y_0, z_0) is

$$J_f(x_0, y_0, z_0) = \begin{pmatrix} 2x_0 & -2y_0 & 0 \\ 2x_0 & 0 & -2z_0 \end{pmatrix}.$$

If $y_0 \neq 0$ and $z_0 \neq 0$, then the extracted 2×2 matrix:

$$d_{(y,z)} f_{(x_0,y_0,z_0)} = \begin{pmatrix} -2y_0 & 0 \\ 0 & -2z_0 \end{pmatrix},$$

is invertible. According to the implicit function theorem, the equation $f(x, y, z) = 0$ has a unique local C^∞ solution near x_0, of the form $(y, z) = \varphi(x) = (\varphi_1(x), \varphi_2(x))$. The differential of φ at x_0 is given by the following formula:

$$d\varphi_{x_0} = -\left[d_{(y,z)} f_{(x_0,y_0,z_0)}\right]^{-1} \circ d_x f_{(x_0,y_0,z_0)},$$

$$J_\varphi(x_0) = -\begin{pmatrix} \frac{\partial f_1}{\partial y}(x_0, y_0, z_0) & \frac{\partial f_1}{\partial z}(x_0, y_0, z_0) \\ \frac{\partial f_2}{\partial y}(x_0, y_0, z_0) & \frac{\partial f_2}{\partial z}(x_0, y_0, z_0) \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{\partial f_1}{\partial x}(x_0, y_0, z_0) \\ \frac{\partial f_2}{\partial x}(x_0, y_0, z_0) \end{pmatrix}.$$

$$J_\varphi(x_0) = -\begin{pmatrix} -2y_0 & 0 \\ 0 & -2z_0 \end{pmatrix}^{-1} \times \begin{pmatrix} 2x_0 \\ 2x_0 \end{pmatrix}.$$

If $x_0 \neq 0$ and $y_0 \neq 0$, then the extracted 2×2 matrix:

$$d_{(x,y)} f_{(x_0,y_0,z_0)} = \begin{pmatrix} 2x_0 & -2y_0 \\ 2x_0 & 0 \end{pmatrix}$$

is invertible. According to the implicit function theorem, the equation $f(x,y,z) = 0$ has a unique local C^∞ solution near z_0, of the form $(x,y) = (\omega_1(z), \omega_2(z))$. The differential of φ at z_0 is given by the following formula:

$$d\varphi_{z_0} = -\left[d_{(x,y)} f_{(x_0,y_0,z_0)}\right]^{-1} \circ d_z f_{(x_0,y_0,z_0)},$$

$$J_\varphi(z_0) = -\begin{pmatrix} \frac{\partial f_1}{\partial x}(x_0,y_0,z_0) & \frac{\partial f_1}{\partial y}(x_0,y_0,z_0) \\ \frac{\partial f_2}{\partial x}(x_0,y_0,z_0) & \frac{\partial f_2}{\partial y}(x_0,y_0,z_0) \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{\partial f_1}{\partial z}(x_0,y_0,z_0) \\ \frac{\partial f_2}{\partial z}(x_0,y_0,z_0) \end{pmatrix}.$$

$$J_\varphi(z_0) = -\begin{pmatrix} 2x_0 & -2y_0 \\ 2x_0 & 0 \end{pmatrix}^{-1} \times \begin{pmatrix} 0 \\ -2z_0 \end{pmatrix}.$$

1.6 Chapter One Exercises

Exercise 1.1.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} y^2 \ln|x|, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

1. Show that f admits a directional derivative in all directions at $(0,0)$.

2. Is the function f differentiable at $(0,0)$?

Exercise 1.2.

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the functions defined by

   $$f(x,y) = \left(e^{2x+y}, 3y - \cos x, x^2 + y + 2\right) \quad \text{and} \quad g(x,y,z) = \left(3x + 2y + z^2, x^2 - z + 1\right).$$

   If, we set $F = f \circ g$ and $G = g \circ f$, then calculate $dF_{(0,0,0)}$ and $dG_{(0,0)}$.

2. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable functions, and let F be defined by:

   $$F(x,y) = f(x,y,g(x,y))$$

   2.1. Express the differential $dF_{(x_0,y_0)}$ in terms of the partial derivatives of f and g.

   2.2. Assume that $F(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. Determine $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in terms of the partial derivatives of f.

Exercise 1.3.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function defined by

$$f(x,y) = (e^x \cos y, e^x \sin y)$$

1. Show that f is a surjective map from $\mathbb{R}^2$ to $\mathbb{R}^2 \setminus \{(0,0)\}$.

2. Let $(x_0, y_0) \in \mathbb{R}^2$. Compute the Jacobian matrix of f at (x_0, y_0) and deduce that f is a local C^1-diffeomorphism in the neighbourhood of (x_0, y_0).

3. Does f realize a global C^1-diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2 \setminus \{(0,0)\}$?

Exercise 1.4.

Let a, $b \in \mathbb{R}$ such that $|ab| < 1$

1. Let $v \in \mathbb{R}$. We define $g : \mathbb{R} \longrightarrow \mathbb{R}$ by $g(x) = x + a\sin(v - b\sin x)$. Show that g is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

2. Define $f(x, y) = (x + a\sin y, y + b\sin x)$. Show that f is a C^1-diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Exercise 1.5.

Consider the function $\phi : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\phi(u, v) = (u + v, uv) \text{ and } U = \left\{(u, v) \in \mathbb{R}^2, \quad u > v\right\}.$$

1. Show that the function ϕ is a C^1-diffeomorphism from U to an open set that needs to be determined.

Exercise 1.6.

Show that the relation $x + y + z + \sin(xyx) = 0$ defines z as a function of x and y in the neighbourhood of the point $(0, 0, 0)$.Then calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at that point by using two different methods.

Exercise 1.7.

Let $\phi : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be the function defined by

$$\phi(x, y) = \sin(x + y) - \ln(x + y)$$

1. Show that there exist two open intervals I_1 and I_2 containing 1 such that the set

$$E = \{(x, y) \in I_1 \times I_2 : \phi(x, y) = \phi(1, 1)\}$$

is the graph of a function φ of class C^1 from I_1 to I_2 with $\varphi(1) = 1$.

2. Give the first-order Taylor expansion of φ in the neighbourhood of 1.

Exercise 1.8.

Let $\phi : GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$, $M \longmapsto M^{-1}$.

1. Prove that ϕ is differentiable at I_n and compute its differential at this point.

2. Answer the same question for an arbitrary $M \in GL_n(\mathbb{R})$.

Exercise 1.9.

Let $n \geq 2$. Consider the determinant function:

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R},\ A \longmapsto \det(A),$$

where $M_n(\mathbb{R})$ denotes the space of real $n \times n$ matrices.

1. Prove that the determinant function is of class C^∞ on $M_n(\mathbb{R})$.

2. For $1 \leq i, j \leq n$, we define

$$f(t) = \det(I_n + tE_{i,j}),$$

where $E_{i,j}$ is the matrix with 1 in the (i, j)-entry and 0 elsewhere. Determine f.

3. Deduce the value of $\frac{\partial \det}{\partial E_{i,j}}(I_n)$.

4. Deduce the expression of the differential of det at I_n.

5. Recover this result by calculating $\det(I_n + tH)$ by diagonalizing H.

6. Prove that if A is invertible, then $d\det_A(H) = \mathrm{tr}(\,(\mathrm{comat}(A))^t H)$.

7. Prove that the previous formula remains valid for any matrix $A \in M_n(\mathbb{R})$.

1.7 Solutions to Chapter One Exercises

Exercise 1.1.

1. Let $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$, we have

$$\lim_{t \longrightarrow 0} \frac{f\left((0,0) + t\nu\right) - f\left((0,0)\right)}{t} = \lim_{t \longrightarrow 0} \frac{f\left(tv_1, tv_2\right)}{t}.$$

If $v_1 = 0$, then

$$\lim_{t \longrightarrow 0} \frac{f\left(tv_1, tv_2\right)}{t} = 0.$$

If $v_1 \neq 0$, then

$$\lim_{t \longrightarrow 0} \frac{f\left(tv_1, tv_2\right)}{t} = \lim_{t \longrightarrow 0} \frac{t^2 v_2^2 \ln|tv_1|}{t} = \lim_{t \longrightarrow 0} tv_2^2 \ln|t| + tv_2^2 \ln|v_1| = 0.$$

Thus $\frac{\partial}{\partial \nu} f\left(0,0\right) = 0$ for all $\nu \in \mathbb{R}^2$.

2. Let $n \in \mathbb{N}^*$, then we consider the sequence $\left(e^{-n^2}, \frac{1}{n}\right)$ which goes to $(0,0)$ when $n \longrightarrow \infty$. Moreover, we have

$$\forall n \in \mathbb{N}^*,\ f\left(e^{-n^2}, \frac{1}{n}\right) = -1 \neq 0.$$

Thus, f is not continuous at $(0,0)$, and therefore not differentiable at this point.

Exercise 1.2.

1. Let us calculate the differential of $F = f \circ g$ at $(0,0,0)$. By the chain rule, we have

$$dF_{(0,0,0)} = df_{g(0,0,0)} \circ dg_{(0,0,0)}.$$

We have $g\left(0,0,0\right) = (0,1)$, so we evaluate $df_{\ (0,1)}$. We get

$$df_{(0,1)} = \begin{pmatrix} 2e & e \\ 0 & 3 \\ 0 & 1 \end{pmatrix},\ dg_{(0,0,0)} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Multiplying:

$$dF_{(0,0,0)} = \begin{pmatrix} 6e & 4e & -e \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{pmatrix}$$

In the same way, we find

$$dG_{(0,0)} = dg_{f(0,0)} \circ df_{(0,0)} = dg_{(1,-1,2)}.df_{(0,0)} = \begin{pmatrix} 6 & 13 \\ 4 & 1 \end{pmatrix}.$$

2. Let us calculate the differential of $F(x,y) = f(x,y,g(x,y))$ at (x_0,y_0). Here $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a scalar function, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are differentiable functions. Its differential at a point (x_0,y_0) is the linear map whose matrix is:

$$dF_{(x_0,y_0)} = \begin{pmatrix} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x} & \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} \end{pmatrix}.$$

where the partial derivatives of f are evaluated at $(x_0,y_0,g(x_0,y_0))$ and those of g at (x_0,y_0). Equivalently, by the chain rule

$$dF_{(x_0,y_0)} = df_{\phi(x_0,y_0)} \circ d\phi_{(x_0,y_0)},$$

where $\phi(x,y) = (x,y,g(x,y)) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the parameterization of the graph surface $z = g(x,y)$, and

$$d\phi_{(x_0,y_0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix}.$$

This matrix describes the tangent plane to the surface parameterized by ϕ.

3. If we assume that $f(x,y,g(x,y)) = 0$ for all (x,y), then this equation defines g implicitly. We differentiate both sides:

   - With respect to x :

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial g}{\partial x} = 0 \implies \frac{\partial g}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial g}{\partial z}}.$$

- With respect to y :

$$\frac{\partial f}{\partial y}+\frac{\partial g}{\partial z}\frac{\partial g}{\partial y}=0 \implies \frac{\partial g}{\partial y}=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial g}{\partial z}}.$$

These are the standard formulas from the implicit function theorem, where the variable $z=g(x,y)$ is implicitly defined by the equation $f(x,y,z)=0$.

Exercise 1.3.

1. Let $(a,b)\in\mathbb{R}^2\backslash\{(0,0)\}$. Define $a'=\frac{a}{\sqrt{a^2+b^2}}$ and $b'=\frac{b}{\sqrt{a^2+b^2}}$. Then $(a',b')\in\mathbb{S}^1$, and there exists $y\in\mathbb{R}$ such that $\cos y=a'$ and $\sin y=b'$. Next, choose $x\in\mathbb{R}$ such that $e^x=\sqrt{a^2+b^2}$. Then $f(x,y)=(a,b)$ and f defines a surjective map from $\mathbb{R}^2$ onto $\mathbb{R}^2\backslash\{(0,0)\}$.

2. The Jacobian matrix of f at (x_0,y_0) is

$$J_f((x_0,y_0))=\begin{pmatrix} e^{x_0}\cos y_0 & -e^{x_0}\sin y_0 \\ e^{x_0}\sin y_0 & e^{x_0}\cos y_0 \end{pmatrix}.$$

Its determinant is e^{2x_0}, which is non-zero. Thus, by the local inversion theorem, f realizes a C^1-diffeomorphism from a neighbourhood U of (x_0,y_0) onto $f(U)$.

3. No, because f is not injective: $f(x,y)=f(x,y+2\pi)$.

Exercise 1.4.

1. To prove this, we first verify that g is continuous (this is clear), and then that g is strictly increasing. Indeed, g is differentiable, and for all $x\in\mathbb{R}$, we have

$$g'(x)=1-ab\cos x\cos(v-b\sin x)>0,$$

since $|ab|<1$. Then, g defines a strictly increasing bijection from $\mathbb{R}$ onto $g(\mathbb{R})$. Now, since $\lim_{x\longrightarrow-\infty}g(x)=-\infty$ and $\lim_{x\longrightarrow+\infty}g(x)=+\infty$, then g defines a bijection from $\mathbb{R}$ onto $\mathbb{R}$.

2. We first observe that f is of class C^1. Its Jacobian matrix is

$$J_f(x,y) = \begin{pmatrix} 1 & a\cos y \\ b\cos x & 1 \end{pmatrix}.$$

and its determinant is

$$\det J_f(x,y) = 1 - ab\cos x\cos y > 0.$$

Therefore, by the Global Inverse Function Theorem, it remains only to prove that f is a bijection from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Let $(u,v) \in \mathbb{R}^2$. Solving $f(x,y) = (u,v)$ is equivalent to the system

$$f(x,y) = (u,v) \Longleftrightarrow \begin{cases} x + a\sin(v - b\sin x) = u \\ y = v - b\sin x \end{cases}.$$

From Question 1, the first equation admits a unique solution $x = g^{-1}(u)$. Substituting this into the second equation yieldsIt

$$y = v - b\sin g^{-1}(u).$$

Hence, for every $(u,v) \in \mathbb{R}^2$, there exists a unique pair $(x,y) \in \mathbb{R}^2$ such that $f(x,y) = (u,v)$. This proves that f is a bijection from $\mathbb{R}^2$ onto $\mathbb{R}^2$. Consequently, f is a C^1-diffeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Exercise 1.5.

1. We begin by determining $\phi(U)$. Let $(u,v) \in U$ and set $(s,p) = \phi(u,v)$. Then $s = u + v$ and $p = uv$, which means that u and v are the two distinct roots of the equation: $x^2 - sx + p = 0$. This quadratic equation must have two distinct real roots. Therefore, its discriminant must be strictly positive, i.e., $s^2 - 4p > 0$.

Now, if $s^2 - 4p > 0$, the equation $\phi(u,v) = (s,p)$ has a unique solution in U. In fact, this solution is

$$u = \frac{s + \sqrt{s^2 - 4p}}{2} \text{ et } v = \frac{s - \sqrt{s^2 - 4p}}{2}.$$

Thus, we have just proven that ϕ is a bijection from U onto $V = \{(s,p) \in \mathbb{R}^2,\ s^2 > 4p\}$, with the inverse bijection

$$(s,p) \longmapsto \left(\frac{s + \sqrt{s^2 - 4p}}{2}, \frac{s - \sqrt{s^2 - 4p}}{2} \right).$$

Since both ϕ and ϕ^{-1} are of class C^1, it follows that ϕ is a C^1-diffeomorphism from U onto V. Moreover, because we obtained an explicit formula for ϕ^{-1}, it is not necessary to invoke the Global Inversion Theorem—although we could have done so, since the Jacobian determinant of ϕ at (u,v) is $u - v > 0$.

Exercise 1.6.

1. Let f be the function defined on $\mathbb{R}^3$ by: $f(x,y,z) = x+y+z+\sin(xyz)$. This function is of class C^∞, it satisfies $f(0,0,0) = 0$, and its partial derivative with respect to z is given by

$$\frac{\partial f}{\partial z}(x,y,z) = 1 + xy\cos(xyz).$$

In particular, $\frac{\partial f}{\partial z}(0,0,0) = 1 \neq 0$, so the implicit function theorem applies. Therefore, there exists a neighbourhood $U \subset \mathbb{R}^2$ of $(0,0)$, a neighbourhood $I \subset \mathbb{R}$ of 0, and a function $\varphi : U \longrightarrow I$ of class C^∞ such that

$$\forall (x,y,z) \in U \times I : f(x,y,z) = 0 \Longleftrightarrow z = \varphi(x,y).$$

We now compute the gradient $\nabla\varphi(0,0)$. By Theorem 1.6, we have

$$\nabla\varphi(0,0) = \left(\frac{\partial\varphi}{\partial x}(0,0), \frac{\partial\varphi}{\partial y}(0,0) \right) = -\frac{\left(\frac{\partial f}{\partial x}(0,0,0), \frac{\partial f}{\partial y}(0,0,0) \right)}{\frac{\partial f}{\partial z}(0,0,0)},$$

which gives

$$\frac{\partial\varphi}{\partial x}(0,0) = \frac{\partial\varphi}{\partial y}(0,0) = -1.$$

Alternatively, we can use the relation

$$x + y + \varphi(x,y) + \sin(xy\varphi(x,y)) = 0,$$

and differentiating it respect to x. We find:

$$1+\frac{\partial\varphi}{\partial x}(x,y)+y\varphi(x,y)\cos(xy\varphi(x,y))+xy\frac{\partial\varphi}{\partial x}(x,y)\cos(xy\varphi(x,y))=0.$$

Evaluating this relation at $(0,0)$ yields

$$\frac{\partial\varphi}{\partial x}(0,0)=-1.$$

The computation with respect to y is entirely analogous and leads to

$$\frac{\partial\varphi}{\partial y}(0,0)=-1.$$

Exercise 1.7.

1. Let

$$\phi(x,y)=\sin(x+y)-\ln(x+y),\ F(x,y)=\phi(x,y),\ c=F(1,1)=\sin 2-\ln 2$$

We consider the level set

$$E=\{(x,y):F(x,y)=c\}.$$

We have

$$\frac{\partial F}{\partial y}(x,y)=\cos(x+y)-\frac{1}{x+y},\ \frac{\partial F}{\partial y}(1,1)=\cos 2-\frac{1}{2}<0.$$

Since the partial derivative with respect to y does not vanish at $(1,1)$, the Implicit Function Theorem guarantees the existence of open intervals I_1, I_2 containing 1 and a function φ of class C^1 from I_1 to I_2 such that:

$$F(x,\varphi(x))=c,\ \varphi(1)=1.$$

So the set E is locally the graph of a function φ of class C^1.

2. The first-order Taylor expansion of φ at $x=1$ is

$$\varphi(x)=\varphi(1)+\varphi'(1)(x-1)+o(x-1).$$

Since

$$\varphi(1) = 1 \text{ and } \varphi'(1) = -\frac{\frac{\partial F}{\partial x}(1,1)}{\frac{\partial F}{\partial y}(1,1)} = -1,$$

then the first-order Taylor expansion at $x = 1$ is:

$$\varphi(x) = 2 - x + o(x-1).$$

Exercise 1.8.

1. To establish differentiability of ϕ at I_n, consider a perturbation $H \in M_n(\mathbb{R})$ such that $I_n + H$ is invertible, which holds for $\|H\|$ sufficiently small. For $\|H\| < 1$, we have

$$(I_n + H)^{-1} = I_n - H + H^2 - H^3 + ...$$

Hence,

$$\phi(I_n + H) = (I_n + H)^{-1} = I_n - H + O\left(\|H\|^2\right),$$

where $O\left(\|H\|^2\right)$ denotes terms of order $\|H\|^2$ or higher as $\|H\| \longrightarrow 0$. Since $\phi(I_n) = I_n^{-1} = I_n$, we obtain

$$\phi(I_n + H) - \phi(I_n) = \left(I_n - H + O\left(\|H\|^2\right)\right) - I_n = -H + O\left(\|H\|^2\right).$$

Thus, the linear part in H is $-H$, and the remainder is $O\left(\|H\|^2\right)$, which is $o(\|H\|)$. Therefore, ϕ is differentiable at I_n, and the differential $d\phi_{I_n}$ is the linear map:

$$d\phi_{I_n} : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R}),\ H \longmapsto -H.$$

2. To establish differentiability of ϕ at arbitrary $M \in GL_n(\mathbb{R})$, consider a perturbation $H \in M_n(\mathbb{R})$ such that $M + H$ is invertible (which holds for $\|H\|$ sufficiently small), write:

$$M + H = M\left(I_n + M^{-1}H\right).$$

Hence,

$$\phi(M + H) = (M + H)^{-1} = \left(I_n + M^{-1}H\right)^{-1} M^{-1}.$$

Using the Neumann series for $(I_n + K)$ with $K = M^{-1}H$ and $\|K\| < 1$, gives

$$\left(I_n + M^{-1}H\right)^{-1} = I_n - M^{-1}H + \left(M^{-1}H\right)^2 - ....$$

Therefore,

$$\phi(M+H) - \phi(M) = \left[I_n - M^{-1}H + O\left(\|H\|^2\right)\right] M^{-1} - M^{-1} = -M^{-1}HM^{-1} + O\left(\|H\|^2\right).$$

The linear part in H is $M^{-1}HM^{-1}$, and the remainder is $O\left(\|H\|^2\right)$, which is $o(\|H\|)$. Therefore, ϕ is differentiable at M, and the differential $d\phi_M$ is the linear map:

$$d\phi_M : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R}),\ H \longmapsto -M^{-1}HM^{-1}.$$

Exercise 1.9.

1. The determinant is a finite sum and product of smooth (actually polynomial) functions. Therefore, $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $A \longmapsto \det(A)$, is of class C^∞.

2. Fix $1 \leq i, j \leq n$ and define $f(t) = \det(I_n + tE_{i,j})$, where $E_{i,j}$ is the matrix with 1 at position (i,j), and 0 elsewhere. So, $I_n + tE_{i,j}$ is the identity matrix with only one perturbed entry: the $(i,j)-$entry is $\delta_{i,j} + t$. Thus

$$f(t) = \begin{cases} 1 + t \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}$$

3. We compute the partial derivative of the determinant at the identity matrix in the direction of the elementary matrix

$$\frac{\partial \det}{\partial E_{i,j}}(I_n) = \lim_{t \longrightarrow 0} \frac{\det(I_n + tE_{i,j}) - \det(I_n)}{t}$$
$$= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} = \delta_{i,j}$$

So, this partial derivative is the Kronecker delta $\delta_{i,j}$.

4. The differential of det at I_n is a linear map from $M_n(\mathbb{R})$ to $\mathbb{R}$. From Question 3, the directional derivative in the direction $E_{i,j}$ is $\delta_{i,j}$. For a general matrix $H = (h_{ij}) \in M_n(\mathbb{R})$, we express H in the basis $\{E_{i,j}\}_{1\leq i,j\leq n}$ as

$$H = \sum_{1\leq i,j\leq n} h_{ij} E_{i,j}.$$

By linearity of the differential, we obtain

$$d\det_{I_n}(H) = \sum_{1\leq i,j\leq n} h_{ij} d_{I_n}(E_{i,j}) = \sum_{1\leq i,j\leq n} h_{ij}\delta_{i,j} = \sum_{1\leq i\leq n} h_{ii} = tr(H).$$

Thus, the differential of det at I_n is

$$d\det_{I_n}(H) = tr(H).$$

5. Assume that H is diagonalizable over $\mathbb{C}$, so $H = PDP^{-1}$ for some invertible matrix P and diagonal matrix $D = diag(\lambda_1, ..., \lambda_n)$. Then $I_n + tH = P(I_n + tD)P^{-1}$, and hence

$$\det(I_n + tH) = \det(I_n + tD) = \prod_{i=1}^{n}(1 + t\lambda_i).$$

Define $g(t) = \prod_{i=1}^{n}(1 + t\lambda_i)$. The derivative of g at $t = 0$ is

$$g'(t) = \sum_{j=1}^{n} \lambda_j \prod_{i\neq j}(1 + t\lambda_i), \quad g'(0) = \sum_{j=1}^{n} \lambda_j .1 = tr(H)$$

Therefore

$$d\det_{I_n}(H) = tr(H).$$

6. Here, comat(A) denotes the cofactor matrix of A, Its transpose, $(\text{comat}(A))^t$, is precisely the matrix adj(A). For any invertible matrix A, adj$(A) = \det(A)A^{-1}$. To compute the differential of the determinant, observe that

$$\det(A + tH) = \det(A)\det\left(I_n + tA^{-1}H\right).$$

For small t, we expand

$$\det\left(I_n + tA^{-1}H\right) = 1 + t\ tr\left(A^{-1}H\right) + o(t).$$

Hence,

$$\begin{aligned}\det(A+tH)-\det(A) &= \det(A)\left[1+t\ tr\left(A^{-1}H\right)+o(t)\right]-\det(A)\\ &= t\ \det(A)\,tr\left(A^{-1}H\right)+o(t).\end{aligned}$$

The linear part in t is $t\ \det(A)\,tr\left(A^{-1}H\right)$. Since $tr\left(A^{-1}H\right)=tr\left(HA^{-1}\right)$, then

$$d\underset{A}{\det}(H)=\det(A)\,tr\left(A^{-1}H\right)=tr\left(\det(A)\,A^{-1}H\right)=tr\left(\text{adj}\,(A)\,H\right).$$

As $\left(\text{comat}\,(A)\right)^{t}=\text{adj}(A)$, it follows that

$$d\underset{A}{\det}(H)=tr(\,(\text{comat}\,(A))^{t}\,H).$$

7. We show that the formula holds for any matrix $A\in M_n(\mathbb{R})$. The partial derivative of $\det(A)$ with respect to the entry (a_{ij}) is the $(i,j)-$cofactor $C_{ij}(A)$ which is the $(i,j)-$entry of the cofactor matrix $\text{comat}(A)$. Thus

$$\frac{\partial\det}{\partial a_{ij}}=C_{ij}(A)$$

The differential in A applied to $H=(h_{ij})$ is

$$\begin{aligned}d\underset{I_n}{\det}(H) &= \sum_{1\leq i,j\leq n}\frac{\partial\det}{\partial a_{ij}}h_{ij}=\sum_{1\leq i,j\leq n}C_{ij}(A)\,h_{ij}\\ &= tr\left(\text{adj}\,(A)\,H\right)=tr(\,(\text{comat}\,(A))^{t}\,H).\end{aligned}$$

Chapter 2

Submanifolds of $\mathbb{R}^n$ and Abstract Manifolds

In this chapter, submanifolds are explored through four equivalent definitions, then tangent and normal spaces, differentiability on submanifolds, and an introduction to abstract manifolds via charts, atlases, and tangent spaces, with worked exercises. Draws on Aime [1], Berger-Gostiaux [3], Tu [7], Viterbo [8], and Waldspurger [9].

2.1 Definitions of a Submanifold of $\mathbb{R}^n$

Let n, $d \in \mathbb{N}^*$, such that $0 \leq d \leq n$. We consider the decomposition:

$$\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}.$$

Definition 2.1. (Original version) A subset M of $\mathbb{R}^n$ is called a C^k submanifold of dimension d if for every point $x \in M$, there exists a neighbourhood U_x of x in $\mathbb{R}^n$, and a C^k-diffeomorphism $\varphi : U_x \longrightarrow \varphi(U_x) \subset \mathbb{R}^n$ that maps x to 0, such that

$$\varphi(U_x \cap M) = \varphi(U_x) \cap \left(\mathbb{R}^d \times \{0_{n-d}\}\right).$$

Remark 2.1. We have the following remarks:

1. The quantity $n-d$ is called the codimension of M.
2. Since φ induces a homeomorphism from $U_a \cap M$ onto the open set $\varphi(U_a) \cap \mathbb{R}^d$ in $\mathbb{R}^d$, then M locally preserves the same topological properties as $\mathbb{R}^d$.

Examples 2.1. We illustrate the definition of a topological submanifold with two extreme cases:

1. If M is a topological submanifold of dimension n, then M is an open set of $\mathbb{R}^n$. In fact: M is of dimension n. Thus, for every $x \in M$, there exists a neighbourhood U_x of x and a homeomorphism $\varphi : U_x \longrightarrow \varphi(U_x)$ that maps x to 0, such that

$$\varphi(U_x \cap M) = \varphi(U_x) \cap \mathbb{R}^n = \varphi(U_x).$$

   Since φ is a homeomorphism, $U_x \cap M = U_x$, which proves that $U_x \subset M$. Hence, M is a neighbourhood of each of its points, which proves that M is an open set in $\mathbb{R}^n$.

2. If M is a topological submanifold of dimension 0, then M is a discrete set. In fact: M is of dimension 0. Thus, for every $x \in M$, there exists a neighbourhood U_x of x and C^∞-diffeomorphism $\varphi : U_x \longrightarrow \varphi(U_x)$ that maps x to 0, such that

$$\varphi(U_x \cap M) = \varphi(U_x) \cap \{0_{\mathbb{R}^n}\} = \{0_{\mathbb{R}^n}\}.$$

   Since φ is a diffeomorphism, $U_x \cap M = \varphi^{-1}(\{0_{\mathbb{R}^n}\}) = \{x\}$. This proves that all points of M are isolated, which proves that M is a discrete set.

Definition 2.2. (Submersion) Let M be a subset of $\mathbb{R}^n$. M is a submanifold of dimension d of $\mathbb{R}^n$ and of class C^k, if and only if, for every $x \in M$, there exists a neighbourhood U_x of x in $\mathbb{R}^n$ and a function $g : U_x \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n-d}$ of class C^k, such that:

1. $g^{-1}(0) = U_x \cap M$.

2. g is a submersion at x (i.e., dg_x is surjective).

Example 2.2. Let's show that $\mathbb{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^∞ using the characterization of the inverse image of a submersion.

Let $x = (x_1, x_2) \in \mathbb{S}_1$. We set $U_x = \mathbb{R}^2 - \{(0,0)\}$, and we consider the map

$$\begin{array}{rccl} g: & U_x & \to & \mathbb{R} \\ & (x_1, x_2) & \mapsto & g(x_1, x_2) = x_1^2 + x_2^2 - 1 \end{array}$$

It is clear that g is of class C^∞, and since $x = (x_1, x_2) \in U_x = \mathbb{R}^2 - \{(0,0)\}$, then we have

$$J_g(x_1, x_2) = (2x_1 \ 2x_2) \neq (0 \ \ 0).$$

Since the Jacobian of g at x has full rank, then g is a submersion at x, and we have

$$g^{-1}(0) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\right\} = S_1 = U_x \cap S_1.$$

This proves that $\mathbb{S}_1$ is a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^∞.

Definition 2.3. (Immersion) Let M be a subset of $\mathbb{R}^n$. M is a submanifold of dimension d of $\mathbb{R}^n$ and class C^k, if and only if, for every $x \in M$, there exists a neighbourhood U_x of x in $\mathbb{R}^n$, a neighbourhood Ω of a in $\mathbb{R}^d$ and a map $\phi : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$ of class C^k, such that:

1. $\phi(a) = x$.

2. $\phi : \Omega \longrightarrow U_x \cap M$ is a homeomorphism.

3. $d\phi_a : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ is injective (i.e., rank $d\phi_a = d$).

Example 2.3. Let's show that $\mathbb{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^∞ using parameterization. Let $(x_1, x_2) \in \mathbb{S}_1$ and ϕ the function defined from $]-\pi, \pi]$ to $\mathbb{R}^2$ by:

$$\begin{array}{rccl} \phi: & \Omega =]-\pi, \pi] & \to & \phi(]-\pi, \pi]) \subset \mathbb{R}^2 \\ & t & \mapsto & \phi(t) = (\cos t, \sin t) \end{array}.$$

ϕ is of class C^∞, and we have:

$$d\phi_t = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

Let $t_0 \in]-\pi, \pi]$. We have $d\phi_{t_0} \neq 0$. Since ϕ is bijective from $]-\pi, \pi]$ onto $\phi(]-\pi, \pi])$, then by the global inversion theorem, ϕ is a C^∞ homeomorphism from Ω onto $\phi(]-\pi, \pi]) = \mathbb{R}^2 \cap \mathbb{S}_1$. Thus by Definition 2.3, $\mathbb{S}_1$ is a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^∞.

Example 2.4. Let's show that

$$\mathbb{C}_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 2t + 1,\ x_2 = \cos t,\ x_3 = \sin t \text{ with } t \in \mathbb{R}\right\}$$

is a submanifold of $\mathbb{R}^3$ of dimension 1 and class C^∞, using parameterization. Let $(x_1, x_2) \in \mathbb{C}_1$ and ϕ the function defined from $\Omega \subset \mathbb{R}$ to $\mathbb{R}^3$ by:

$$\begin{array}{rccl} \phi: & \Omega & \to & \phi(\Omega) \subset \mathbb{R}^3 \\ & t & \mapsto & \phi(t) = (2t + 1, \cos t, \sin t) \end{array}.$$

1. The function ϕ is of class C^∞ on Ω because each of its components is of class C^∞.

2. The function ϕ is surjective from Ω onto $\phi(\Omega)$ by construction.

3. Now, before proving that ϕ is injective, we first determine the domain Ω on which ϕ be an immersion. We have

$$d\phi_t = \begin{pmatrix} 2 \\ -\sin t \\ \cos t \end{pmatrix}.$$

For ϕ to be an immersion, the rank of $d\phi_t$ must be maximal (i.e., 1 in this case) at every point $t \in \Omega$. This is equivalent to ensuring that at least one of the determinants of $d\phi_t$ (for 1×1 submatrices) is nonzero. Since $\Delta_1 = 2 \neq 0$, then ϕ is an immersion on $\mathbb{R}$, so we can take $\Omega = \mathbb{R}$. For the injectivity of ϕ on Ω, we can easily show that for all t_1 and t_2 in Ω, we have $\phi(t_1) = \phi(t_2)$ implies $t_1 = t_2$, therefore ϕ is injective from Ω to $\phi(\Omega)$.

4. It is clear that the function

$$\begin{array}{rccl} \phi^{-1}: & \phi(\Omega) & \to & \Omega \\ & (x_1,x_2,x_3) & \mapsto & \phi^{-1}(x_1,x_2,x_3)=\frac{x_1-1}{2} \end{array}.$$

is of class C^0 on $\phi(\Omega)$. Finally, if we take $U_x=\mathbb{R}^3$, then:

$$\phi(\Omega)=\mathbb{C}_1\cap\mathbb{R}^3=\mathbb{C}_1.$$

Definition 2.4. (Graph of a function) A subset M of $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ of dimension d if and only if, for every $x=\left(\overline{x},\overline{\overline{x}}\right)\in M\subset\mathbb{R}^d\times\mathbb{R}^{n-d}$, there exists an open set U_x in $\mathbb{R}^n$ containing x, another open se $V_{\overline{x}}$ in $\mathbb{R}^d$ containing $\overline{x}$ and a map $h: V_{\overline{x}}\longrightarrow\mathbb{R}^{n-d}$ of class C^k such that (possibly after permuting the coordinates of x), $U_x\cap M$ is the graph of the mapping h.

$$U_x\cap M=\{(\overline{x},h(\overline{x})),\ \ \overline{x}\in V_{\overline{x}}\}.$$

Example 2.5. Let's show that $\mathbb{S}_1=\{(x_1,x_2)\in\mathbb{R}^2: x_1^2+x_2^2=1\}$ is a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^∞ using the characterization of a manifold as the graph of a function:

$$\begin{array}{rccl} U_1=\{x_1\in\,]-1,1[\,,\ x_2>0\} & & & \\ h_1: \ ]-1,1[& \to & \mathbb{R} & \\ x_1 & \mapsto & x_2=+\sqrt{1-x_1^2} & \end{array}\quad,\quad \begin{array}{rccl} U_3=\{x_2\in\,]-1,1[\,,\ x_1>0\} & & & \\ h_3: \ ]-1,1[& \to & \mathbb{R} & \\ x_2 & \mapsto & x_1=+\sqrt{1-x_2^2} & \end{array}.$$

$$\begin{array}{rccl} U_2=\{x_1\in\,]-1,1[\,,\ x_2<0\} & & & \\ h_2: \ ]-1,1[& \to & \mathbb{R} & \\ x_1 & \mapsto & x_2=-\sqrt{1-x_1^2} & \end{array}\quad,\quad \begin{array}{rccl} U_4=\{x_2\in\,]-1,1[\,,\ x_1<0\} & & & \\ h_4: \ ]-1,1[& \to & \mathbb{R} & \\ x_2 & \mapsto & x_1=-\sqrt{1-x_2^2} & \end{array}.$$

Thus

$$\begin{array}{ll} U_1\cap\mathbb{S}_1=\{(x_1,h_1(x_1)),\ x_1\in\,]-1,1[\}, & U_3\cap\mathbb{S}_1=\{(x_2,h_3(x_2)),\ x_2\in\,]-1,1[\} \\ U_2\cap\mathbb{S}_1=\{(x_1,h_2(x_1)),\ x_1\in\,]-1,1[\}, & U_4\cap\mathbb{S}_1=\{(x_2,h_4(x_2)),\ x_2\in\,]-1,1[\} \end{array}.$$

2.2 Equivalence Between Definitions of a Submanifold

Theorem 2.1. The four previous definitions of a submanifold are equivalent.

Proof. Def 2.1 $\Longrightarrow$ Def 2.2: Assume that M satisfies Definition 2.1 and let us show that it satisfies Definition 2.2. Let $x \in M$. By Definition 2.1, there exists a neighbourhood of $U \subset \mathbb{R}^n$ of x , a neighbourhood of $V \subset \mathbb{R}^n$ of 0, and a C^k-diffeomorphism $\varphi : U \longrightarrow V$, such that

$$\varphi(M \cap U) = V \cap \left(\mathbb{R}^d \times \{0_{n-d}\}\right).$$

Denote by $pr_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-d}$ to the projection onto the last $n-d$, coordinates and define

$$g = pr_2 \circ \varphi : U \longrightarrow \mathbb{R}^{n-d}.$$

Since φ is a diffeomorphism, its differential $d\varphi_x$ is bijective. Therefore,

$$dg_x(\mathbb{R}^n) = pr_2(d\varphi_x(\mathbb{R}^n)) = pr_2(\mathbb{R}^n) = \mathbb{R}^{n-d},$$

which shows that g is a submersion at x.

It remains to verify that $M \cap U = g^{-1}(\{0\})$. For any $x' \in M \cap U$, we have

$$\varphi(x') \in \varphi(M \cap U) = V \cap \left(\mathbb{R}^d \times \{0_{n-d}\}\right) \subset \mathbb{R}^d \times \{0_{n-d}\},$$

hence $pr_2 \circ \varphi(x') = 0$, i.e. $g(x') = 0$.

Conversely, if $x' \in g^{-1}(\{0\})$, then $pr_2(\varphi(x')) = 0$ so $\varphi(x') \in \mathbb{R}^d \times \{0_{n-d}\}$. Since $x' \in U$, we have $\varphi(x') \in V$, hence

$$\varphi(x') \in V \cap \left(\mathbb{R}^d \times \{0_{n-d}\}\right) = \varphi(M \cap U),$$

and therefore $x' \in M \cap U$.

Def 2.2 $\Longrightarrow$ Def 2.4**:** Suppose that M satisfies Definition 2.2, and let us show that it also satisfies Definition 2.4. Let $x \in M$, By Definition 2.2, there exists a neighbourhood $U \subset \mathbb{R}^n$ of x and a C^k map $g : U \longrightarrow \mathbb{R}^{n-d}$ which is a submersion at x, such that

$$M \cap U = g^{-1}(\{0\}).$$

Choose an orthonormal basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ such that

$$span\{dg_x(e_{d+1}), ..., dg_x(e_n)\} = \mathbb{R}^{n-d}$$

Such a basis exists because $dg_x : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-d}$ is surjective.

From now on, we use the coordinate system defined by this basis. In this system, we write

$$x = (x_1, ..., x_n)$$

The derivative of g with respect to $(x_{d+1}, ..., x_n)$ is a surjective map from $\mathbb{R}^{n-d}$ onto $\mathbb{R}^{n-d}$, hence bijective. Therefore, by the Implicit Function Theorem, there exist a neighbourhood $U' \subset U$ of x, a neighbourhood V of $(x_1, ..., x_d)$, and a C^k map $h : V \longrightarrow \mathbb{R}^{n-d}$ such that

$$U' \cap g^{-1}(\{0\}) = \{(t, h(t)),\ t \in V\}.$$

Thus, $M \cap U' = U' \cap g^{-1}(\{0\})$ =graph(h).

Def 2.4 $\Longrightarrow$ Def 2.3: Assume that M satisfies Definition 2.4, and let us show that it also satisfies Definition 2.3. Let $x \in M$. By translating coordinates, we may assume that $x = 0$, which simplifies the notation. Then there exists a neighbourhood $U \subset \mathbb{R}^n$ of $x = 0$, an open set $V \subset \mathbb{R}^d$, and a C^k map $h : V \longrightarrow \mathbb{R}^{n-d}$ such that, in a suitable coordinate system,

$$M \cap U = \text{graph}(h) = \{(t, h(t)),\ t \in V\}.$$

Define

$$\begin{array}{rccc} f: & V & \to & \mathbb{R}^n \\ & t & \mapsto & (t, h(t)). \end{array}$$

This is a C^k map. It is an immersion at 0, since for every $t \in \mathbb{R}^d$,

$$df_0(t) = (t_1, ..., t_d, dh(0)(t)),$$

which can vanish only if $t = 0$.

We also have $f(0) = 0 = x$ (indeed $h(0) = 0$ because $x = 0 \in M \cap U$ =graph(h)), and f defines a homeomorphism between V and $f(V)$ (whose inverse is simply the projection onto the first d coordinates). Consequently,

$$M \cap U = \text{graph}(h) = f(V).$$

Def 2.3 $\Longrightarrow$ Def 2.1: Suppose that M satisfies Definition 2.3, and let us show that it also satisfies Definition 2.1. Let $x \in M$. Then there exist neighbourhoods U, V of x and 0 in $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively, and a C^k map $f : V \longrightarrow \mathbb{R}^n$ that is a homeomorphism from V onto $f(V)$, such that

$$M \cap U = f(V),$$

and f is an immersion at a, where a is the unique preimage of 0 by f. By translating coordinates, we may assume, for simplicity of notation, that $a = 0$, i.e. $f(0) = x$.

By the local form of immersions theorem (Theorem 1.1), there exists a neighbourhood $V' \subset V$ of $0_{\mathbb{R}^d}$ and a C^k-diffeomorphism $\varphi : A \longrightarrow B$ between a neighbourhood A of x and a neighbourhood B of $0_{\mathbb{R}^n}$, such that

$$\forall (t_1, ...t_d) \in V',\ \varphi \circ f(t_1, ...t_d) = (t_1, ...t_d, 0, ..., 0).$$

Now, choose a neighbourhood $E \subset A \cap U$ of x such that

- $f^{-1}(f(V) \cap E) \subset V'$ (such a neighbourhood exists because f is a homeomorphism onto its image, so f^{-1} is well defined and continuous on $f(V)$),
- $\varphi(E) \subset V' \cap \mathbb{R}^{n-d}$.

Set $F = \varphi(E)$. Then φ is a C^k-diffeomorphism from E onto F. We now show that

$$\varphi(M \cap E) = F \cap \mathbb{R}^d \times \{0_{n-d}\}.$$

Indeed, if $(t_1, ...t_d, 0, ..., 0) \in \mathbb{R}^d \times \{0_{n-d}\} \cap F$, then $t := (t_1, ...t_d)$ belongs to V' (since $F = \varphi(E) \subset V' \cap \mathbb{R}^{n-d}$). Thus

$$(t_1, ...t_d, 0, ..., 0) = \varphi \circ f(t).$$

But $f(t) \in f(V) \subset M$ and $f(t) \in \varphi^{-1}(F) = E$. This shows that

$$(t_1, ...t_d, 0, ..., 0) = \varphi(M \cap E).$$

Hence, we obtain the inclusion $\varphi(M \cap E) \supset \mathbb{R}^d \times \{0_{n-d}\} \cap F$, which completes the proof.

2.3 Tangent Space of a Submanifold

Definition 2.5. Let M be a submanifold of dimension d of $\mathbb{R}^n$ and of class C^k. A vector $v \in \mathbb{R}^n$ is called a tangent vector to M at x if there exists a curve $C : I =]-\delta, +\delta[\longrightarrow \mathbb{R}^n$, of class C^1 such that:

1. $C(0) = x$, i.e. the curve passes through x,

2. $C(t) \in M$ for all $t \in I$, i.e. the curve lies entirely on M,

3. $C'(0) = \frac{dC}{dt}(0) = v$, i.e. v is the velocity vector of C at $t = 0$.

Examples 2.6. We illustrate the notion of tangent vectors to a submanifold by considering curves lying on it.

1. The unit cercle in $\mathbb{R}^2$: Let $\mathbb{S}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$. Consider the curve

$$\begin{array}{rccc} C_1 : & I =]-\frac{\pi}{2}, \frac{\pi}{2}[& \to & \mathbb{R}^2 \\ & t & \mapsto & C(t) = (\cos t, \sin t) \end{array},$$

   for every $t \in I$, we have $C_1(t) \in \mathbb{S}_1$. In particular $C_1(0) = (1, 0)$. The derivative (velocity) at 0 is

$$C_1'(0) = v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2. The unit sphere in $\mathbb{R}^3$: Let $\mathbb{S}_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$. Consider the curve

$$\begin{array}{rccc} C_2 : & I =]-\frac{\pi}{2}, \frac{\pi}{2}[& \to & \mathbb{R}^3 \\ & t & \mapsto & C_2(t) = (\cos t, \sin t, 0) \end{array},$$

   for every $t \in I$, we have $C_2(t) \in \mathbb{S}_2$. In particular $C_2(0) = (1, 0, 0)$. The derivative (velocity) at 0 is

$$C_2'(0) = v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Definition 2.6. The tangent space, denoted T_xM, is the set of all tangent vectors to M at x.

Proposition 2.1. T_xM is a vector space of the same dimension as the submanifold M.

Theorem 2.2. Let M be a submanifold of dimension d of $\mathbb{R}^n$ and class C^k, and let $x \in M$.

1. If U and V are neighbourhoods of x and 0 in $\mathbb{R}^n$, respectively, and if $\varphi : U \longrightarrow V$ is C^k-diffeomorphism such that $\varphi(x) = 0$ and $\varphi(M \cap U) = V \cap (\mathbb{R}^d \times \{0_{n-d}\})$, then
$$T_xM = [d\varphi_x]^{-1} \left(\mathbb{R}^d \times \{0_{n-d}\}\right).$$

2. If there exists an open neighbourhood U of x, such that $M \cap U = g^{-1}\{0\}$, where $g : U \longrightarrow \mathbb{R}^{n-d}$ is a submersion at x, then:
$$T_xM = \ker dg_x.$$

3. If there exists a parameterization (Ω, ϕ) of $U \cap M$, where U is a neighbourhood of x and $\phi(a) = x$, then
$$T_xM = (d\phi_a)\left(\mathbb{R}^d\right) = \mathrm{Im}\,(d\phi_a).$$

4. If U is a neighbourhood of x, V an open subset of $\mathbb{R}^d$ and $h : V \longrightarrow \mathbb{R}^{n-d}$ a C^k map such that, in a suitable coordinate system, $M \cap U =$graph(h), then
$$T_xM = \left\{\left(t_1, ..., t_d, dh_{(x_1,...,x_d)}(t_1, ..., t_d)\right),\ t_1, ..., t_d \in \mathbb{R}\right\}.$$

Proof. See the proof in [9].

Example 2.7. Let $x = (x_1, x_2, x_3) \in \mathbb{S}_2$. We set $U_x = \mathbb{R}^3 - \{(0,0,0)\}$, and we consider the map
$$\begin{array}{rccl} g: & U_x & \to & \mathbb{R} \\ & (x_1, x_2, x_3) & \mapsto & g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1 \end{array}$$
It is clear that g is of class C^∞, and since $x = (x_1, x_2, x_3) \in U_x = \mathbb{R}^3 - \{(0,0,0)\}$, then we have
$$J_g(x_1, x_2, x_3) = (2x_1\ 2x_2\ 2x_3) \neq (0\ \ 0\ \ 0).$$

Since the Jacobian of g at x has full rank, then g is a submersion at x, and we have

$$g^{-1}(0) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\} = \mathbb{S}_2 = U_x \cap \mathbb{S}_2.$$

This proves that $\mathbb{S}_2$ is a submanifold of $\mathbb{R}^3$ of dimension 2 and of class C^∞. Let us calculate $T_{(0,0,1)}\mathbb{S}_2$. We have

$$\begin{aligned} T_{(0,0,1)}\mathbb{S}_2 = \ker dg_{(0,0,1)} &= \left\{H = (h_1, h_2, h_3) \in \mathbb{R}^3 : dg_{(0,0,1)}(H) = 0\right\} \\ &= \left\{H = (h_1, h_2, h_3) \in \mathbb{R}^3 : 2h_3 = 0\right\} = \langle (1,0,0), (0,1,0) \rangle . \end{aligned}$$

2.4 Tangent Bundle

Definition 2.7. Let M be a submanifold of $\mathbb{R}^n$ of dimension d . The tangent bundle of M is the set

$$TM = \bigsqcup_{x \in M} \{x\} \times T_x M = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M,\ v \in T_x M\}.$$

Proposition 2.2. If M is a submanifold of $\mathbb{R}^n$ of class C^k $(k \geq 2)$ and of dimension d, then TM is a C^{k-1} submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ of dimension $2d$.

2.5 Normal Space and Normal Bundle

Definition 2.8. Let M be a d dimensional C^k submanifold of $\mathbb{R}^n$. The normal space to M at x is the subspace $N_x M$ of $\mathbb{R}^n$ consisting of all vectors orthogonal orthogonal to $T_x M$ with respect to the standard inner product on $\mathbb{R}^n$, i.e

$$N_x M = (T_x M)^\perp .$$

Proposition 2.3. Let M be a d dimensional C^k submanifold of $\mathbb{R}^n$ and let $x \in M$. The normal space $N_x M$ is a linear subspace of $\mathbb{R}^n$ of dimension $n - d$. Moreover:

- If $f = (f_1, ..., f_{n-k}) : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n-d}$ is a regular equation for M in a neighbourhood of x, then

$$N_x M = span\left(\nabla f_1(x), ..., \nabla f_{n-k}(x)\right).$$

- If $g : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$ satisfies $g(\Omega) \subset M$, $g(y) = x$, and dg_y is injective, then

$$N_x M = \left\{ v \in \mathbb{R}^n : \left\langle v, \frac{\partial g}{\partial x_i}(y) \right\rangle = 0 \text{ for all } i = 1, ..., d \right\}.$$

Definition 2.9. Let M be a d dimensional C^k submanifold of $\mathbb{R}^n$. The normal bundle of M is the set

$$NM = \bigsqcup_{x \in M} \{x\} \times N_x M = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M,\ v \in N_x M\}.$$

Proposition 2.4. If M is a C^k $(k \geq 2)$ submanifold of $\mathbb{R}^n$ of dimension d, then NM is a C^{k-1} submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ of dimension n.

2.6 Differentiability on Submanifolds

Definition 2.10. Let $M_1 \subset \mathbb{R}^n$ and $M_2 \subset \mathbb{R}^m$ be submanifolds, and let $f : M_1 \longrightarrow M_2$ be a map. We say that f is differentiable at $x \in M_1$ if, for every local parameterization $\varphi : U \subset \mathbb{R}^d \longrightarrow M_1$ with $x \in \varphi(U) \subset M_1$, the function $f \circ \varphi : U \longrightarrow \mathbb{R}^m$ is differentiable at $y = \varphi^{-1}(x)$. We say that f is differentiable on M_1 if it is differentiable at every point x in M_1.

It is straightforward to verify that this definition does not depend on the choice of the local parametrization (φ, U).

Lemma 2.1. Let $M_1 \subset \mathbb{R}^n$ and $M_2 \subset \mathbb{R}^m$ be submanifolds. A map $f : M_1 \longrightarrow M_2$ is differentiable if and only if it is locally the restriction of a differentiable map $F : V \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Definition 2.11. Let M_1 and M_2 be submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, of class at least C^k $(k \geq 1)$. A map $\psi : M_1 \longrightarrow M_2$ is called a C^k-diffeomorphism if it is a home-

omorphism and ψ and ψ^{-1} are of class C^k. In this case, we say that M_1 and M_2 are C^k-diffeomorphic.

Definition 2.12. Let $M_1 \subset \mathbb{R}^n$ and $M_2 \subset \mathbb{R}^m$ be submanifolds, and let $f : M_1 \longrightarrow M_2$ be a differentiable map. The differential of f at $x \in M_1$ is the linear map

$$\begin{array}{rccl} d_x f: & T_x M_1 & \to & T_{f(x)} M_2 \\ & v & \mapsto & d_x f\,(v) = d_{f(x)} F\,(v) \end{array}$$

where $F : V \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is any local differentiable extension of f.

2.7 Abstract Manifolds

2.7.1 Topological Space, Chart and Atlas

Definition 2.13 Let M be a non-empty set, and let τ be a collection of subsets of M. Suppose the following conditions hold:

1. $\emptyset \in \tau$ and $M \in \tau$.
2. The union of any collection of elements of τ is an element of τ.
3. The intersection of any finite collection of elements of τ is an element of τ.

Then τ is called a topology on M. The pair (M, τ) is called a topological space, and the elements of τ are called open sets.

Definition 2.14 A topological space (M, τ) is said to be separated or Hausdorff if, for any two distinct points $x, y \in M$ with $x \neq y$, there exist two open sets U_x and U_y such that:

$$x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

Definition 2.15. A topological manifold of dimension d is a topological space (M, τ) that satisfies the following conditions:

1. (M, τ) is separated (Hausdorff).

2. For every point $x \in M$, there exists an open set U_x in M containing x , and a homeomorphism φ from U_x onto an open subset of $\mathbb{R}^d$.

The pair (U_x, φ) is called a chart on M.

Definition 2.16. Let M be a topological manifold of dimension d. A family of charts $(U_i, \varphi_i)_{i \in J}$, indexed by a set J, is an index set, is called an atlas of M of dimension d if the open sets $(U_i)_{i \in J}$ form a cover of M, i.e.,

$$\bigcup_{i \in J} U_i = M.$$

Definition 2.17. Let M be a topological manifold. Suppose (U, φ) and (V, ψ) are two charts such that $U \cap V = \emptyset$. The map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is called a chart transition map or transition function.

Definition 2.18. An atlas $(U_i, \varphi_i)_{i \in J}$ of topological manifold M is said to be of class C^k, if all the transition functions

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$

are of class C^k.

Proposition 2.5. The following statements hold:

1. The union of two atlases on M is an atlas on M of class C^k if and only if all charts in the union are mutually C^k-compatible.

2. Two atlases on M of class C^k are said to be C^k-equivalent if their union is an atlas on M of class C^k

3. The union of all atlases that belong to an equivalence class under the C^k-equivalence relation is the largest atlas in that class, which is called the maximal atlas.

Definition 2.19. Let M be a topological manifold of dimension d. We say that M is a differentiable manifold of dimension d and class C^k if it is equipped with a maximale atlas of dimension d and class C^k.

Theorem 2.3. If A is an atlas of class C^k on a topological space M, then there exists a unique maximal atlas $A_{\max}$ containing A consisting of all charts that are compatible with those in A.

Definition 2.20. Let X be a topological space, and let $\sim$ be an equivalence relation on X. The topological quotient $X/\sim$ is the set of equivalence classes

$$X/\sim = \{[x]\,,\ x \in X\}\,,$$

endowed with the quotient topology, defined as follows:

A subset $U \subset X/\sim$ is open if and only if $\pi^{-1}(U) \subset X$ is open, where

$$\pi : X \longrightarrow X/\sim,\ x \longmapsto [x]$$

is the canonical projection.

2.7.2 Tangent Space of an Abstract Manifold

Definition 2.21 (Tangent space via curves). Let M be an abstract manifold and let $p \in M$. The tangent space to M at the point p, denoted T_pM is defined using smooth curves through p as follows:

1. Smooth Curves through p: Consider the set of smooth maps:

$$\gamma :]-\epsilon, \epsilon[\longrightarrow M \text{ such that } \gamma(0) = p.$$

2. Equivalence Relation: Two curves γ and δ through p are said to be equivalent at p, written $\gamma \sim \delta$, if there exists a local chart (U, φ) with $p \in U$ such that:

$$\frac{d}{dt}_{|t=0} (\varphi \circ \gamma)(t) = \frac{d}{dt}_{|t=0} (\varphi \circ \delta)(t)\,.$$

This relation is independent of the choice of chart.

3. Definition of the Tangent Space: The tangent space at p is the set of equivalence classes of smooth curves through p under the relation $\sim$:

$$T_pM = \{[\gamma]\,,\ \gamma :]-\epsilon, \epsilon[\longrightarrow M,\ \gamma(0) = p\}\,.$$

Each class $[\gamma] \in T_pM$ is called a tangent vector to M at p.

Example 2.8. Define

$$\mathbb{RP}^1 = \left(\mathbb{R}^2 - \{(0,0)\}\right) / \sim,\ \ (x,y) \sim (\lambda x, \lambda y)\,,\ \lambda \in \mathbb{R}^*.$$

Let $\pi : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{RP}^1$ be the natural projection. We equip $\mathbb{RP}^1$ the quotient topology:

$$U \subset \mathbb{RP}^1 \text{ is open if and only if } \pi^{-1}(U) \text{ is open in } \mathbb{R}^2 - \{(0,0)\}\,.$$

In this example we will:

1) Show that $U_1 = \{[x:y] \in \mathbb{RP}^1,\ x \neq 0\}$, $U_2 = \{[x:y] \in \mathbb{RP}^1,\ y \neq 0\}$ are open and cover $\mathbb{RP}^1$.

2) Show that $\mathbb{RP}^1$ is Hausdorff.

3) Define the charts: $\varphi_1 : U_1 \longrightarrow \mathbb{R}$, $\varphi_1([x:y]) = \frac{y}{x}$, $\varphi_2 : U_2 \longrightarrow \mathbb{R}$, $\varphi_2([x:y]) = \frac{x}{y}$, compute $\varphi_2 \circ \varphi_1^{-1}$, and check it is C^∞.

4) Conclude that $\mathbb{RP}^1$ is a 1-dimensional differentiable manifold.

Recall: the quotient topology on $\mathbb{RP}^1$.

The real projective line $\mathbb{RP}^1$ is defined as the set of all lines through the origin in the plane $\mathbb{R}^2$. In other words, we consider the set $\mathbb{R}^2 - \{(0,0)\}$, and we identify two nonzero vectors if they are scalar multiples of each other. Formally, we define an equivalence relation:

$$(x,y) \sim (\lambda x, \lambda y) \text{ for all } \lambda \in \mathbb{R}^*.$$

The resulting quotient space, denoted $\mathbb{RP}^1$, consists of equivalence classes of this relation. Each point in $\mathbb{RP}^1$represents a direction in the plane and can be denoted using homogeneous coordinates $[x : y]$, representing the class of a nonzero pair (x, y).

A subset $U \subset \mathbb{RP}^1$is open if and only if its preimage under the canonical projection

$$\pi : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{RP}^1, \ \ (x, y) \longmapsto [x : y]$$

is an open set of $\mathbb{R}^2 \setminus \{(0,0)\}$. In other words:

$$U \text{ is open in } \mathbb{RP}^1 \iff \pi^{-1}(U) \text{ is open in } \mathbb{R}^2 \setminus \{(0,0)\}.$$

We consider the two subsets

$$U_1 = \left\{[x : y] \in \mathbb{RP}^1, \ x \neq 0\right\}, \ U_2 = \left\{[x : y] \in \mathbb{RP}^1, \ y \neq 0\right\}.$$

Let us show that these subsets are open in the quotient topology:

· For U_1, the preimage $\pi^{-1}(U_1) = \{(x, y) \in \mathbb{R}^2, \ x \neq 0\}$ is open in $\mathbb{R}^2 \setminus \{(0,0)\}$ as it's the complement of the closed y-axis.

· Similarly, $\pi^{-1}(U_2) = \{(x, y) \in \mathbb{R}^2, \ y \neq 0\}$ is open in $\mathbb{R}^2 \setminus \{(0,0)\}$ as it's the complement of the closed x-axis.

These subsets cover $\mathbb{RP}^1$ because any point $[x : y]$ must have at least one non-zero coordinate (since $(0, 0)$ is excluded). Thus every equivalence class is represented in either U_1 or U_2 (or both).

Define the charts

$$\varphi_1 : U_1 \longrightarrow \mathbb{R}, \ \varphi_1([x : y]) = \frac{y}{x},$$
$$\varphi_2 : U_2 \longrightarrow \mathbb{R}, \ \varphi_2([x : y]) = \frac{x}{y},$$

To prove $\mathbb{RP}^1$ is Hausdorff, consider two distinct points $[x_1 : y_1]$ and $[x_2 : y_2]$. We examine two cases:

Case 01. both $x_1 \neq 0$ and $x_2 \neq 0$: We can write them as $[1 : t_1]$ and $[1 : t_2]$ where $t_1 = \frac{y_1}{x_1}$, $t_2 = \frac{y_2}{x_2}$. Since the points are distinct, $t_1 \neq t_2$. Take disjoint intervals I_1, I_2 around t_1, t_2 in $\mathbb{R}$. Then $\varphi_1^{-1}(I_1)$ and $\varphi_1^{-1}(I_2)$ are disjoint open sets separating the points.

Case 02. both $x_1 = 0$ and $x_2 \neq 0$: Represent them as $[0 : 1]$ and $[1 : t_2]$. We can find:

· A neighbourhood of $[0 : 1]$ where $\left|\frac{y}{x}\right| > M$.

· A neighbourhood of $[1 : t_2]$ where $\left|\frac{y}{x} - t_2\right| < \epsilon$.

These are disjoint for appropriate M and ϵ.

Define the charts

$$\varphi_1 : U_1 \longrightarrow \mathbb{R},\ \varphi_1([x : y]) = \frac{y}{x},\ \text{with inverse } \varphi_1^{-1}(t) = [1 : t]$$
$$\varphi_2 : U_2 \longrightarrow \mathbb{R},\ \varphi_2([x : y]) = \frac{x}{y},\ \text{with inverse } \varphi_2^{-1}(s) = [s : 1]$$

The transition map on $U_1 \cap U_2$ is:

$$\varphi_2 \circ \varphi_1^{-1}(t) = \varphi_2([1 : t]) = \frac{1}{t}.$$

This map is of class C^∞ on $\mathbb{R}^*$, and so is its inverse. Therefore, the charts are compatible, and the manifold structure is well-defined. Consequently, $\mathbb{RP}^1$ is a smooth abstract manifold of dimension 1.

2.8 Chapter Two Exercises

Exercise 2.1.

Determine whether each of the following functions is an immersion or a submersion:

1. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $(x, y) \longmapsto (x, y, 0)$.
2. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $(x, y, z) \longmapsto (y, z)$.
3. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(x, y, z) \longmapsto xy + 2yz + 3xz$.
4. $f : \mathbb{R} \longrightarrow \mathbb{R}^2$, $t \longmapsto (\sin(2t), \sin(3t))$.
5. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $(x, y, z) \longmapsto (x^2 + y^2 + z^2, xy)$.
6. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $(x, y) \longmapsto (e^x, \cos(y), \sin(y))$.

Exercise 2.2.

Determine which of the following sets are submanifolds and give their dimension.

1. $M_1 = \{(x, y, z) \in \mathbb{R}^3;\ z = x - 2(x^2 + y^2)\}$.
2. $M_2 = \{(t, t^2);\ t \in \mathbb{R}\}$.
3. $M_3 = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\}$.
4. $M_4 = \{(x, y) \in \mathbb{R}^2;\ xy = 0\}$.
5. $M_5 = \left\{(x, y, z) \in \mathbb{R}^3;\ x^2 + 2y^2 + z^2 = 1 \text{ and } \left(x - \frac{1}{2}\right)^2 + y^2 - z^2 = 0\right\}$.

Exercise 2.3.

For which values of $\alpha \in \mathbb{R}$, the set $M_\alpha = \{(x, y) \in \mathbb{R}^2;\ y^2 - x^4 = \alpha\}$ is a submanifold of $\mathbb{R}^2$?

Exercise 2.4.

Determine which of the following sets are submanifolds of $\mathbb{R}^2$ and give their dimension.

1. $V_1 = \{(t^2, t^3)\,;\ t > 0\}$.

2. $V_2 = \{(t^2, t^3)\,;\ t \geq 0\}$.

3. $V_3 = \{(t^2, t^3)\,;\ t \in \mathbb{R}\}$.

Exercise 2.5.

Let M_1 be a submanifold of $\mathbb{R}^n$ of dimension d_1, and M_2 a submanifold of $\mathbb{R}^m$ of dimension d_2. Define their product by

$$M_1 \times M_2 = \left\{x = (x_1, x_2) \in \mathbb{R}^{n+m};\ x_1 \in M_1,\ x_2 \in M_2\right\}.$$

1. Show that $M_1 \times M_2$ is a submanifold of $\mathbb{R}^{n+m}$, and determine its dimension.

2. Let $a = (a_1, a_2) \in M_1 \times M_2$. Show that the tangent space satisfies:

$$T_a\left(M_1 \times M_2\right) = T_{a_1}M_1 \times T_{a_2}M_2 = \{(t_1, t_2)\,,\ t_1 \in T_{a_1}M_1,\ t_2 \in T_{a_2}M_2\}.$$

Exercise 2.6.

Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$\forall\,(x, y, z) \in \mathbb{R}^3,\ f\,(x, y, z) = x^2 + y^2 - z^2 - 1,$$

and let $M = f^{-1}\left(\{0\}\right)$, the level set corresponding to a hyperboloid.

1. Compute $\nabla f\,(x, y, z)$.

2. Show that there exists an open neighbourhood $U \subset \mathbb{R}^3$ of $(1, 1, 1)$, an open neighbourhood $V \subset \mathbb{R}^2$ of $(1, 1)$, and a smooth function $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$f\,(x, y, z) = 0 \Leftrightarrow z = \varphi\,(x, y).$$

3. Determine the set $E \subset \mathbb{R}^3$ where f is a submersion.

4. Conclude that M is a 2-dimensional submanifold of $\mathbb{R}^3$?

5. Find an equation of the tangent space to M at the point $(1,1,1)$.

Exercise 2.7.

Let $A \in M_n(\mathbb{R})$ be an invertible symmetric matrix such that the associated bilinear form $x \in \mathbb{R}^n \longmapsto \langle Ax, x\rangle$ is positive definite. Let $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We set

$$\mathbb{S} = \{x \in \mathbb{R}^n : \langle Ax, x\rangle + \langle b, x\rangle + c = 0\},$$

and we define the discriminant by:

$$\Delta = \left\langle A^{-1}b, b\right\rangle - 4c.$$

1. Discuss the geometric nature of the set $\mathbb{S}$ based on the sign of Δ.

2. Show that for any $x \in \mathbb{S}$ the tangent space is $T_x\mathbb{S} = \{v \in \mathbb{R}^n : \langle 2Ax + b, v\rangle = 0\}$.

Exercise 2.8.

Let $n \geq 1$ and $\mathbb{O}_n(\mathbb{R})$ the set defined by:

$$\mathbb{O}_n(\mathbb{R}) = \left\{M \in M_n(\mathbb{R}) : \ M^tM = I\right\}.$$

1. Prove that $\mathbb{O}_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$ with dimension $\frac{n(n-1)}{2}$.

2. Show that the tangent space at I is $T_I\mathbb{O}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) : \ M^t = -M\}$.

Exercise 2.9.

Let D be the set of all lines in the Euclidean plane $\mathbb{R}^2$. Define the subsets U and V of D as follows:

$$U = \{d \in D \mid d : y = ax + b, \ a, b \in \mathbb{R}\},$$

$$V = \{d \in D \mid d : x = a'y + b', \ a', b' \in \mathbb{R}\}.$$

1. Show that $D = U \cup V$.

2. Define a topology on D by declaring that a subset $W \subseteq D$ is open if and only if: $\varphi(W \cap U)$ and $\psi(W \cap V)$ are open in $\mathbb{R} \times \mathbb{R}$, where the charts (U, φ) and (V, ψ) are defined by:

$$\varphi : U \longrightarrow \mathbb{R} \times \mathbb{R},\ \varphi(d) = (a, b) \text{ if } d : y = ax + b,$$

$$\psi : V \longrightarrow \mathbb{R} \times \mathbb{R},\ \psi(d') = (a', b') \text{ if } d' : x = a'y + b'.$$

   Show that with this topology, φ and ψ are homeomorphisms onto their images.

3. Show that D is Hausdorff.

4. Calculate explicitly the transition maps $\varphi \circ \psi^{-1}$and $\psi \circ \varphi^{-1}$, and verify that both are of class C^∞.

5. Deduce that D is a 2-dimensional differentiable abstract manifold of class C^∞.

Exercise 2.10.

Let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by

$$x \sim y \Longleftrightarrow x - y \in 2\pi\mathbb{Z}.$$

Let $\mathbb{R}/(2\pi\mathbb{Z})$ denote the set of equivalence classes, and let $\pi : \mathbb{R} \longrightarrow \mathbb{R}/(2\pi\mathbb{Z})$ be the canonical projection sending $x \longmapsto [x]$.

1. Define the quotient topology on $\mathbb{R}/(2\pi\mathbb{Z})$ induced by π. Show that $\mathbb{R}/(2\pi\mathbb{Z})$, endowed with this topology, is a Hausdorff space.

2. Define two local charts on $\mathbb{R}/(2\pi\mathbb{Z})$ using the intervals $]0, 2\pi[$ and $]-\pi, \pi[$. Show that these charts are homeomorphisms onto open subsets of $\mathbb{R}$.

3. Determine the overlap between the two charts, and show that the transition maps between them are of class C^∞.

4. Deduce that $\mathbb{R}/(2\pi\mathbb{Z})$ is a 1-dimensional differentiable abstract manifold.

2.9 Solutions to Chapter Two Exercises

Exercise 2.1.

Let us first note that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, its Jacobian matrix at X, $J_f(X)$ maps $\mathbb{R}^n$ to $\mathbb{R}^p$. It can only be injective if $n \leq p$ and surjective if $n \geq p$. In other words, the condition $n \leq p$ is necessary for f to be an immersion, and the condition $n \geq p$ is necessary for f to be a submersion. We then compute the Jacobian matrix J_f for each function and verify whether it is injective or surjective at each point.

1. We have
$$J_f(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
This matrix is always of rank 2 (the dimension of the domain of f), so f is an immersion (but not a submersion).

2. We have
$$J_f(x, y, z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
This matrix is always of rank 2 (the dimension of the codomain of f), so f is a submmersion (but not an immersion).

3. We have
$$J_f(x, y, z) = \begin{pmatrix} y + 3z & x + 2z & 2y + 3x \end{pmatrix}.$$
This matrix is identically zero if $x = y = z = 0$. The function f is neither an immersion nor a submersion.

4. We have
$$J_f(t) = \begin{pmatrix} \cos(2t) \\ \cos(3t) \end{pmatrix}.$$

This matrix is of rank 1 for all $t \in \mathbb{R}$. Indeed, it is not possible to have both $\cos(2t) = 0$ and $\cos(3t) = 0$ simultaneously. Otherwise, there would exist k, $l \in \mathbb{Z}$ such that $t = \frac{\pi}{4} + \frac{k\pi}{2}$ and $t = \frac{\pi}{6} + \frac{l\pi}{3}$. This leads to the equality $2l - 3k = (1/2)$ which is impossible since the left-hand side is an integer. Thus, f is an immersion, but not a submersion.

5. We have
$$J_f(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ y & x & 0 \end{pmatrix}.$$
This matrix is identically zero if $x = y = z = 0$. The function f is neither an immersion nor a submersion.

6. We have
$$J_f(x,y) = \begin{pmatrix} e^x & 0 \\ 0 & -\sin(y) \\ 0 & \cos(y) \end{pmatrix}.$$
This matrix is of rank 2 for all $(x,y) \in \mathbb{R}^2$. Indeed, $e^x \neq 0$ and the vector $\begin{pmatrix} -\sin(y) \\ \cos(y) \end{pmatrix}$ cannot simultaneously be zero. Thus, f is an immersion, but not a submersion.

Exercise 2.2.

1. M_1 is the graph of the function $h_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x,y) \longmapsto x - 2(x^2 + y^2)$. Therefore, it is a submanifold (of dimension 2) of $\mathbb{R}^3$.

2. M_2 is also a submanifold. For instance, it is the graph of the function $h_2 : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$. It is a submanifold (of dimension 1) of $\mathbb{R}^2$.

3. Let $f_1(x,y,z) = x^2 + y^2 + z^2 - 1$, so that $M_3 = f^{-1}(\{0\})$. Then f_1 is a submersion at every point of M_3. Indeed, $df_{1_{(x,y,z)}} = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$, and $df_{1_{(x,y,z)}}$ is surjective except when $x = y = z = 0$, but this point is not in M_3. Thus, M_3 is a submanifold (of dimension 2) of $\mathbb{R}^3$.

4. Assume, for contradiction, that M_4 is a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^k. By Definition 2.1, there exists a neighbourhood W of $(0,0)$ in $\mathbb{R}^2$ and a C^1-diffeomorphism $\varphi : W \longrightarrow \varphi(W)$ such that

$$\varphi(M_4 \cap W) = \varphi(W) \cap \{\mathbb{R} \times \{0\}\}.$$

Removing the origin, $(M_4 \cap W) \setminus \{(0,0)\}$ has 4 connected components (because $M_4 = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ near the origin consists of the two coordinate axes, each with two sides), since φ is a homeomorphism, it induces a homeomorphism

$$\varphi : (M_4 \cap W) \setminus \{(0,0)\} \longrightarrow \varphi(W) \cap \{\mathbb{R} \times \{0\}\} \setminus \{\varphi(0,0)\}$$

Hence the right-hand set must also have 4 connected components. But $\{\mathbb{R} \times \{0\}\} \setminus \{\varphi(0,0)\}$ (an open interval in the line with one point removed) has exactly 2 connected components. This is a contradiction. Therefore M_4 is not a submanifold of $\mathbb{R}^2$ of dimension 1 and class C^k.

Exercise 2.3.

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by: $f(x,y) = y^2 - x^4 - \alpha$, and consider the level set: $M_\alpha = f^{-1}(\{0\})$. The differential of f is given by:

$$df_{(x,y)} = \begin{pmatrix} -4x^3 & 2y \end{pmatrix},$$

which vanishes only at$(x,y) = (0,0)$. Thus f is a submersion at all points except $(0,0)$. We distinguish the two cases:

Case 1: $\alpha \neq 0$, in this case, $(0,0) \notin M_\alpha$, and f is a submersion at every point of M_α. Hence M_α is a submanifold of dimension 1 in $\mathbb{R}^2$.

Case 2: $\alpha = 0$, in this case, $(0,0) \in M_0$. Assume for contradiction that M_0 is a submanifold of dimension 1 in $\mathbb{R}^2$. Then there exists a neighbourhood W of $(0,0)$ in $\mathbb{R}^2$ and a C^1-diffeomorphism $\varphi : W \longrightarrow \varphi(W)$ such that

$$\varphi(M_0 \cap W) = \varphi(W) \cap \{\mathbb{R} \times \{0\}\}$$

But $(M_0 \cap W) \setminus \{(0,0)\}$ has 4 connected components (the two branches $y = x^2$ and $y = -x^2$), while the set $\varphi(W) \cap \{\mathbb{R} \times \{0\}\} \setminus \{\varphi(0,0)\}$ is an open interval in a line with one point removed, hence has only 2 connected components. Contradiction.

Exercise 2.4.

1. V_1 is a $1-$dimensional submanifold of $\mathbb{R}^2$. The simplest way to establish this is by using a parametrization. Consider the function:

$$\begin{array}{rccl} g: &]0,+\infty[& \to & V_1 \\ & t & \mapsto & g(t) = (t^2, t^3) \end{array}.$$

The function g is a smooth bijection (i.e., of class C^∞), defined on the open set $]0,+\infty[$, and its differential is

$$dg_t = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix},$$

which is injective everywhere since this derivative is never the zero vector (as $t \neq 0$).

2. V_2 is not a $1-$dimensional submanifold. The difference from the previous case is that V_2 has an extreme point, namely the point $(0,0)$. Therefore, we cannot have a function $f :]-a,+a[\longrightarrow V_2$ such that $f(0) = (0,0)$ and that is injective. Specifically, if $f(a/2) = (t_0^2, t_0^3)$ with $t_0 > 0$, then by the intermediate value theorem, we have

$$f(]0, a/2[) \supset \left\{ \left(t^2, t^3\right),\ t \in]0, t[\right\}.$$

But if $f(-a/2) = (t_1^2, t_1^3)$ with $t_1 > 0$, then

$$f(]-a/2, 0[) \supset \left\{ \left(t^2, t^3\right),\ t \in]0, t_1[\right\}.$$

The two sets $f(]0, a/2[)$ and $f(]-a/2, 0[)$ cannot be disjoint as they should be.

3. V_3 is not a submanifold either. The issue this time is that $(0,0)$ is a cusp point of V_3, and a submanifold cannot have cusp points. To prove this precisely in our case, suppose that V_3 admits a parametrization by a function f of class C^1 defined on a

neighbourhood $]-a,+a[$ of $0 \in \mathbb{R}$, which gives a bijection between $]-a,+a[$ and $U \cap V_3$, where U is a neighbourhood of $(0,0)$ in $\mathbb{R}^2$, and df_0 is injective. Write $f = (f_1, f_2)$. Then, $f_1(x) > 0$ if $x \neq 0$, hence necessarily $f_1'(0) = 0$. But $f_2'(0) = 0$ as well since we have $f_2 = \pm f_1^{3/2}$. Thus

$$df_0 = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

which contradicts the injectivity of the differential.

Exercise 2.5.

1. Let $a = (a_1, a_2) \in M_1 \times M_2$. There exists a neighbourhood U_1 of a_1 in $\mathbb{R}^n$ and a submersion $f_1 : U_1 \longrightarrow \mathbb{R}^{n-d_1}$ at a_1 such that $M_1 \cap U_1 = f_1^{-1}(\{0\})$. Similarly, there exists a neighbourhood U_2 of a_2 in $\mathbb{R}^m$ and a submersion $f_2 : U_2 \longrightarrow \mathbb{R}^{m-d_2}$ at a_2 such that $M_2 \cap U_2 = f_2^{-1}(\{0\})$.We then define

$$\begin{array}{rccl} f = (f_1, f_2): & U_1 \times U_2 \subset \mathbb{R}^{n+m} & \to & \mathbb{R}^{n+m-d_1-d_2} \\ & (x, y) & \mapsto & (f_1(x), f_2(y)) \end{array}.$$

The function f is a submersion at $a = (a_1, a_2)$. Indeed, its differential at (a_1, a_2) is given by the block matrix

$$df_{(a_1,a_2)} = \begin{pmatrix} df_{1a_1} & 0 \\ 0 & df_{2a_2} \end{pmatrix}.$$

The rank of this matrix is the rank of df_{1a_1} plus the rank of df_{2a_2}, that is, $(n-d_1)+(m-d_2) = (n+m)-(d_1+d_2)$. Moreover,

$$f^{-1}(\{0\}) = (M_1 \cap U_1) \times (M_2 \cap U_2) = (M_1 \times M_2) \cap (U_1 \times U_2).$$

Hence, $M_1 \times M_2$ is a submanifold of $\mathbb{R}^{n+m}$ of dimension $d_1 + d_2$.

Exercise 2.6.

1. The gradient of f is

$$\nabla f(x, y, z) = \begin{pmatrix} 2x & 2y & -2z \end{pmatrix}.$$

2. We have $\frac{\partial f}{\partial z}(1,1,1) = 2 \neq 0$. By the Implicit Function Theorem, there exists a neighbourhood U of $(1,1,1)$ in $\mathbb{R}^3$, a neighbourhood V of $(1,1)$ in $\mathbb{R}^2$, and a smooth function φ from $\mathbb{R}^2$ to $\mathbb{R}$ such that

$$\forall (x,y,z) \in U : f(x,y,z) = 0 \Longleftrightarrow z = \varphi(x,y).$$

3. We have

$$\nabla f(x,y,z) = \begin{pmatrix} 2x & 2y & -2z \end{pmatrix} = (0,0,0) \Longleftrightarrow (x,y,z) = (0,0,0).$$

So for all $(x,y,z) \neq (0,0,0)$, the rank of $\nabla f(x,y,z)$ is 1, which is the dimension of the codomain. Thus, f is a submersion on $E = \mathbb{R}^3 \backslash (0,0,0)$.

4. Since $f \in C^\infty$ and $M = f^{-1}(\{0\})$, it follows from Definition 2.2**,** that, M is a submanifold of $\mathbb{R}^3$ of dimension 2 and of class C^∞.

5. The tangent space at $(1,1,1)$ is:

$$T_{(1,1,1)}M = \ker \nabla f(1,1,1) = \left\{(x,y,z) \in \mathbb{R}^3 : x + y - z = 0\right\}.$$

Exercise 2.7.

1. The set $\mathbb{S}$ is defined by:

$$\mathbb{S} = \{x \in \mathbb{R}^n : \langle Ax, x\rangle + \langle b, x\rangle + c = 0\},$$

where $A \in M_n(\mathbb{R})$ is an invertible symmetric matrix such that the bilinear form $x \longmapsto \langle Ax, x\rangle$ is positive definite, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. The discriminant Δ is:

$$\Delta = \left\langle A^{-1}b, b\right\rangle - 4c.$$

To determine if $\mathbb{S}$ is a submanifold of $\mathbb{R}^n$, we analyze the three cases based on the sign of Δ. A subset $\mathbb{S} \subset \mathbb{R}^n$ is a submanifold if, for every point $x \in \mathbb{S}$, there exists a neighbourhood $U \subset \mathbb{R}^n$ of x and a diffeomorphism $\phi : U \longrightarrow V \subset \mathbb{R}^n$ such that:

$$\phi(\mathbb{S} \cap U) = V \cap \left(\mathbb{R}^d \times \{0_{n-d}\}\right),$$

for some d, or if $\mathbb{S}$ is empty.

Case $\Delta > 0$: In this case $\mathbb{S}$ is non-empty and bounded. The function $f(x) = \langle Ax, x\rangle + \langle b, x\rangle + c$ has the gradient $\nabla f(x) = 2Ax + b$. For all $x \in \mathbb{S}$, $\nabla f(x) \neq 0$ (since $\Delta > 0$ implies the critical point $x_0 = -\frac{1}{2}A^{-1}b$ is not in $\mathbb{S}$). Hence By Definition 2.2, $\mathbb{S} = f^{-1}(0)$ is a smooth embedded submanifold of dimension $n-1$.

Given that A is positive definite, $\mathbb{S}$ is an ellipsoid (compact and connected).

Case $\Delta = 0$: In this case $\mathbb{S} = \{x_0\}$ where $x_0 = -\frac{1}{2}A^{-1}b$. This is a single point. If we take a neighbourhood U of x_0 such that $\mathbb{S} \cap U = \{x_0\}$, and define $\phi : U \longrightarrow \mathbb{R}^n$ by $\phi(x) = x - x_0$, then ϕ is a diffeomorphism, and $\phi(\mathbb{S} \cap U) = \{0\} = \phi(U) \cap (\mathbb{R}^0 \times \{0_n\})$. Thus, $\mathbb{S}$ is a submanifold of dimension 0.

Case $\Delta < 0$: In this case $\mathbb{S} = \emptyset$. The empty set is a submanifold by convention, as the definition holds vacuously.

2. For any $x \in \mathbb{S}$, the tangent space $T_x\mathbb{S}$ is defined as the set of all vectors $v \in \mathbb{R}^n$ such that the directional derivative of the defining function f at x in the direction v is zero. Define:

$$f(x) = \langle Ax, x\rangle + \langle b, x\rangle + c,$$

so $\mathbb{S} = f^{-1}(0)$. The differential $df_x : \mathbb{R}^n \longrightarrow \mathbb{R}$ is given by:

$$df_x(v) = \langle \nabla f(x), v\rangle .$$

Since A is symmetric, the gradient is:

$$\nabla f(x) = 2Ax + b.$$

Thus,

$$df_x(v) = \langle 2Ax + b, v\rangle .$$

The tangent space $T_x\mathbb{S}$ is the kernel of df_x, i.e., the set of v where $df_x(v) = 0$:

$$T_x\mathbb{S} = \{v \in \mathbb{R}^n : \langle 2Ax + b, v\rangle = 0\} .$$

This holds for all $x \in \mathbb{S}$, regardless of the sign of Δ.

Exercise 2.8.

1. Consider the map

$$f : M_n(\mathbb{R}) \longrightarrow S_n(\mathbb{R}),\ f(M) = M^t M - I$$

where $S_n(\mathbb{R})$ is the vector space of symmetric $n \times n$ matrices. Then:

$$\mathbb{O}_n(\mathbb{R}) = \left\{M \in M_n(\mathbb{R}) :\ M^t M = I\right\} = f^{-1}(0).$$

To show $\mathbb{O}_n(\mathbb{R})$ is a submanifold, we use Definition 2.2, and we prove that df_M is surjective. For all $H \in M_n(\mathbb{R})$, we have

$$f(M+H) - f(M) = M^t H + H^t M + H^t H.$$

The linear part (ignoring higher-order terms) gives:

$$df_M(H) = M^t H + H^t M \in S_n(\mathbb{R}).$$

Now, we prove that df_M is surjective for all $M \in \mathbb{O}_n(\mathbb{R})$. For any $S \in S_n(\mathbb{R})$, we need $H \in M_n(\mathbb{R})$ such that $df_M(H) = S$. Set $H = \frac{1}{2}MS$, then

$$\begin{aligned} df_M(H) &= M^t\left(\frac{1}{2}MS\right) + \left(\frac{1}{2}MS\right)^t M \\ &= \frac{1}{2}M^t MS + \frac{1}{2}SM^t M \\ &= \frac{1}{2}S + \frac{1}{2}S = S. \end{aligned}$$

Thus, df_M is surjective. By Definition 2.2, $\mathbb{O}_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$ of dimension $\frac{n(n-1)}{2}$. Indeed

$$\dim \mathbb{O}_n(\mathbb{R}) = \dim M_n(\mathbb{R}) - \dim S_n(\mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

2. The tangent space of $\mathbb{O}_n(\mathbb{R})$ at I is

$$\begin{aligned} T_I\mathbb{O}_n(\mathbb{R}) &= \{H \in M_n(\mathbb{R}) :\ df_I(H) = 0\} \\ &= \left\{H \in M_n(\mathbb{R}) :\ H + H^t = 0\right\}. \end{aligned}$$

Thus

$$T_I\mathbb{O}_n(\mathbb{R}) = \left\{M \in M_n(\mathbb{R}) :\ M^t = -M\right\}.$$

Exercise 2.9.

1. Let us show that $D = U \cup V$. The set D consists of all lines in the Euclidean plane $\mathbb{R}^2$. The subsets can be defined by

$$U = \{d \in D \mid d : y = ax + b,\ a, b \in \mathbb{R}\},$$
$$V = \{d \in D \mid d : x = a'y + b',\ a', b' \in \mathbb{R}\}.$$

A line in $\mathbb{R}^2$ is either vertical, horizontal, or oblique (neither vertical nor horizontal).

• If a line is vertical, it is given by $x = c$ for some $c \in \mathbb{R}$. This can be written as $x = 0.y + c$, so it is in V with $a' = 0$, $b' = c$. Vertical lines are not in U because they cannot be expressed in the form $y = ax + b$ (the slope is undefined).

• If a line is horizontal, it is given by $y = k$ for some $k \in \mathbb{R}$. This can be written as $y = 0.y + k$, so it is in U with $a = 0$, $b = k$. Horizontal lines are not in V because they cannot be expressed in the form $x = a'y + b'$ (x is not a function of y).

• If a line is oblique, it has a defined non-zero slope. It can be written as $y = ax + b$ with $a \neq 0$ (so in U) and as $x = a'y + b'$ with $a' \neq 0$ by solving for $x : x = \frac{1}{a}y - \frac{b}{a}$, so $a' = \frac{1}{a}$, $b' = -\frac{b}{a}$ (thus in V).

Every line in D is either vertical (in V but not U), horizontal (in U but not V), or oblique (in both U and V). Therefore, $D = U \cup V$.

2. Let us show that φ and ψ are homeomorphisms onto their images. The topology on D is defined by declaring a subset $W \subseteq D$ is open iff $\varphi(W \cap U)$ and $\psi(W \cap U)$ are open in $\mathbb{R}^2$, where

$$\varphi : U \longrightarrow \mathbb{R}^2,\ \varphi(d) = (a, b) \text{ if } d : y = ax + b,$$
$$\psi : V \longrightarrow \mathbb{R}^2,\ \psi(d') = (a', b') \text{ if } d' : x = a'y + b'.$$

First, φ and ψ are bijective:

• φ maps U bijectively onto $\mathbb{R}^2$because each non-vertical line has a unique representation $y = ax + b$.

• ψ maps V bijectively onto $\mathbb{R}^2$because each non-horizontal line has a unique representation $x = a'y + b'$.

To show $\varphi : U \longrightarrow \mathbb{R}^2$ is a homeomorphism (with U having the subspace topology from D):

• **Continuity**: Let $O \subseteq \mathbb{R}^2$ be open. Consider $W = \varphi^{-1}(O) \subseteq U \subseteq D$. Then

$\varphi(W \cap U) = \varphi(W) = O$ (open in $\mathbb{R}^2$)

$\psi(W \cap V) = \psi(\varphi^{-1}(O) \cap V) = \{(a', b') \mid (a, b) \in O,\ a \neq 0\}$ with $a' = \frac{1}{a}$, $b' = -\frac{b}{a}$. This is open in $\mathbb{R}^2$ because the map $(a, b) \longmapsto \left(\frac{1}{a}, -\frac{b}{a}\right)$ is a diffeomorphism from $\mathbb{R}^2 \setminus \{a = 0\}$ onto $\mathbb{R}^2 \setminus \{a' = 0\}$, and $O \cap \{a \neq 0\}$ is open in $\mathbb{R}^2 \setminus \{a = 0\}$. Thus, W is open in D, so $\varphi^{-1}(O)$ is open in U.

• **Openness:** Let $A \subseteq U$ be open in U. Then $A = U \cap W'$ for some open W' in D. Then $\varphi(A) = \varphi(U \cap W')$ which is open in $\mathbb{R}^2$ by the definition of the topology.

Since φ is continuous, bijective, and open, it is a homeomorphism onto $\mathbb{R}^2$. Similarly $\psi : V \longrightarrow \mathbb{R}^2$, it is a homeomorphism onto $\mathbb{R}^2$.

3. Let us show that D is Hausdorff. Consider distinct lines $d_1, d_2 \in D$. Cases:

Case 01: Both in U**.** Since U is open (as $\varphi(U \cap U) = \mathbb{R}^2$ open, $\psi(U \cap V) = \mathbb{R}^* \times \mathbb{R}$ open) and $\varphi : U \longrightarrow \mathbb{R}^2$ is a homeomorphism with $\mathbb{R}^2$ Hausdorff, there are disjoint open sets $A_1, A_2 \subseteq U$ separating d_1, d_2. Since U is open in D then d_1, d_2 are open in D.

Case 02: Both in V. Similarly, V is open (as $\varphi(V \cap U) = \mathbb{R}^* \times \mathbb{R}$ open, $\psi(V \cap V) = \mathbb{R}^* \times \mathbb{R}$ open) and $\psi : V \longrightarrow \mathbb{R}^2$ is a homeomorphism with $\mathbb{R}^2$ Hausdorff, there are disjoint open sets $A_1, A_2 \subseteq V$ separating d_1, d_2. Since V is open in D then d_1, d_2 are open in D.

Case 03: One in $U \setminus V$, the other in $V \setminus U$. Suppose $d_1$vertical (in $V \setminus U$), $d_2$horizontal (in $U \setminus V$). Let $\varphi(d_1) = (0, c)$, $\varphi(d_2) = (0, k)$. Choose $\epsilon > 0$, set $O_1 = \{(a', b') : |a'| < \epsilon,\ |b' - c| < \epsilon\} \subseteq \mathbb{R}^2$, $O_2 = \{(a, b) : |a| < \delta,\ |b - k| < \delta\} \subseteq \mathbb{R}^2$ for $\delta < \frac{1}{\epsilon}$. Then:

• $W_1 = \psi^{-1}(O_1)$ is open in D (as shown in Question 2), contains d_1.

• $W_2 = \psi^{-1}(O_2)$ is open in D, contains d_2.

• $W_1 \cap W_2 = \emptyset$: If $d \in W_1 \cap W_2$, then d has parameters $|a'| < \epsilon$, $|b' - c| < \epsilon$ and $|a| < \delta$, $|b - k| < \delta$. Since $a' = \frac{1}{a}$, $|a'| < \epsilon$ implies $|a| > \frac{1}{\epsilon} > \delta > |a|$, contradiction. Thus, D is Hausdorff.

4. Let us calculate transition maps and verify smoothness. The transition maps are defined on $U \cap V$ (lines neither horizontal nor vertical). We have

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \varphi(U \cap V),\ \ (a, b) \longmapsto \left(\tfrac{1}{a}, -\tfrac{b}{a}\right),$$

$$\text{with } \varphi(U \cap V) = \mathbb{R}^* \times \mathbb{R} \text{ (since } a \neq 0).$$

and

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \psi(U \cap V),\ \ (a', b') \longmapsto \left(\tfrac{1}{a'}, -\tfrac{b'}{a'}\right),$$

$$\text{with } \psi(U \cap V) = \mathbb{R}^* \times \mathbb{R} \text{ (since } a \neq 0).$$

Both maps are of class C^∞ because Rational functions with non-vanishing denominators ($a \neq 0$, $a' \neq 0$).

5. D is a 2-dimensional differentiable abstract manifold of class C^∞ because:

• D is covered by the two charts (U, φ) and (V, ψ), each mapping to $\mathbb{R}^2$, so it is locally Euclidean of dimension 2.

• The topology is Hausdorff (Question 3).

• The transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are C^∞ (Question 4), so the atlas $\{(U, \varphi), (V, \psi)\}$ is C^∞ compatible.

Therefore, D is a 2-dimensional differentiable abstract manifold of class C^∞.

Exercise 2.10.

1. Quotient Topology and Hausdorff Property. The quotient topology on $\mathbb{R}/(2\pi\mathbb{Z})$ is defined as follows: a subset $U \subseteq \mathbb{R}/(2\pi\mathbb{Z})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}$, where $\pi : \mathbb{R} \longrightarrow \mathbb{R}/(2\pi\mathbb{Z})$ is the canonical projection sending $x \longmapsto [x]$, and $[x]$ denotes the equivalence class of x under the relation defined on $\mathbb{R}$ by

$$x \sim y \Longleftrightarrow x - y \in 2\pi\mathbb{Z}.$$

To show that $\mathbb{R}/(2\pi\mathbb{Z})$ is Hausdorff, consider two distinct equivalence classes $[x]$, $[y] \in \mathbb{R}/(2\pi\mathbb{Z})$. Since $[x] \neq [y]$, $x - y \notin 2\pi\mathbb{Z}$. Define:

$$\delta = \min_{k\in\mathbb{Z}} |x - y - 2\pi k| > 0,$$

which exists and is positive because the set $\{|x - y - 2\pi k|,\ k \in \mathbb{Z}\}$ is discrete and closed. Set $\epsilon = \frac{\delta}{3} > 0$, and consider the open intervals:

$$A =]x - \epsilon, x + \epsilon[\,,\ B =]y - \epsilon, y + \epsilon[\,,$$

in $\mathbb{R}$. The sets $\pi(A)$ and $\pi(B)$ are open in the quotient topology because:

$$\pi^{-1}(\pi(A)) = \bigcup_{k\in\mathbb{Z}} (A + 2\pi k)\,,\quad \pi^{-1}(\pi(B)) = \bigcup_{k\in\mathbb{Z}} (B + 2\pi k)\,,$$

are unions of open sets in $\mathbb{R}$, hence open. Moreover, $\pi(A) \cap \pi(B) = \emptyset$. Suppose, for contradiction, that there exists $[z] \in \pi(A) \cap \pi(B)$. Then there are k, $m \in \mathbb{Z}$ such that $z \in (A + 2\pi k) \cap (B + 2\pi m)$, so $z = x' + 2\pi k = y' + 2\pi m$ for some $x' \in A$ and $y' \in B$. Thus

$$|x - y - 2\pi(m - k)| \leq |x - x'| + |x' - y'| + |y' - y| < \epsilon + 0 + \epsilon = 2\epsilon = \frac{2\delta}{3} < \delta,$$

contradicting the minimality of δ. Therefore, $\pi(A)$ and $\pi(B)$ are disjoint open neighbourhoods of $[x]$ and $[y]$, respectively, proving that $\mathbb{R}/(2\pi\mathbb{Z})$ is Hausdorff.

2. Local Charts and Homeomorphisms. Define the intervals $U_1 =]0, 2\pi[$ and $U_2 =]-\pi, \pi[$, both open in $\mathbb{R}$ The corresponding subsets of $\mathbb{R}/(2\pi\mathbb{Z})$ are:

$$V_1 = \pi(U_1) = \{[x],\ x \in]0, 2\pi[\},\ V_2 = \pi(U_2) = \{[y],\ y \in]-\pi, \pi[\}.$$

Define the charts:

$$\varphi_1 : V_1 \longrightarrow U_1,\ \varphi_1([x]) = x \text{ for } x \in]0, 2\pi[,$$
$$\varphi_2 : V_2 \longrightarrow U_2,\ \varphi_2([y]) = y \text{ for } y \in]-\pi, \pi[.$$

These maps are well-defined because each equivalence class in V_1 (resp. V_2) has a unique representative in U_1 (resp. U_2).

• Injectivity: If $\varphi_1([x]) = \varphi_1([y])$, then $x = y$, so $[x] = [y]$. Similarly for φ_2.

• Surjectivity: $\varphi_1(V_1) = U_1$, $\varphi_2(V_2) = U_2$, both open in $\mathbb{R}$.

• Continuity: For φ_1, let $W \subseteq U_1$ be open. Then $\varphi_1^{-1}(W) = \pi(W)$, and $\pi^{-1}(\pi(W)) = \bigcup_{k\in\mathbb{Z}} (W + 2\pi k)$ is open in $\mathbb{R}$, so $\pi(W)$ is open in the quotient topology, and thus $\varphi_1^{-1}(W)$ is open in V_1. Similarly for φ_2.

• Openness: For A open in V_1, $A = V_1 \cap W$ for some open W in $\mathbb{R}/(2\pi\mathbb{Z})$. Then $\varphi_1(A) = \pi^{-1}(W) \cap U_1$, which is open in U_1. Similarly for φ_2.

Additionally, V_1 and V_1 are open in $\mathbb{R}/(2\pi\mathbb{Z})$ because:

$$\pi^{-1}(V_1) = \bigcup_{k\in\mathbb{Z}} (U_1 + 2\pi k) = \mathbb{R} \setminus 2\pi\mathbb{Z},\quad \pi^{-1}(V_2) = \bigcup_{k\in\mathbb{Z}} (U_2 + 2\pi k) = \mathbb{R} \setminus (\pi + 2\pi\mathbb{Z}),$$

both open in $\mathbb{R}$. Thus, φ_1 and φ_2 are homeomorphisms onto their images.

3. Transition maps. The overlap is $V_1 \cap V_2$. The images under the charts are:

$$\varphi_1(V_1 \cap V_2) =]0, \pi[\cup]\pi, 2\pi[,\ \varphi_2(V_1 \cap V_2) =]-\pi, 0[\cup]0, \pi[.$$

The transition map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(V_1 \cap V_2) \longrightarrow \varphi_2(V_1 \cap V_2)$ is given by:

$$\varphi_2 \circ \varphi_1^{-1}(x) = \begin{cases} x \text{ if } x \in]0, \pi[\\ x - 2\pi \text{ if } x \in]\pi, 2\pi[\end{cases}.$$

The transition map $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(V_1 \cap V_2) \longrightarrow \varphi_1(V_1 \cap V_2)$ is given by:

$$\varphi_1 \circ \varphi_2^{-1}(x) = \begin{cases} x + 2\pi \text{ if } x \in]-\pi, 0[\\ x \text{ if } x \in]0, \pi[\end{cases}.$$

Both transition maps are of class C^∞ (smooth) because they are either the identity or a translation on each connected component, and these operations are smooth.

4. We conclude that $\mathbb{R}/(2\pi\mathbb{Z})$ is a 1-dimensional differentiable abstract manifold.

Chapter 3

Differential Forms and Integration

In this chapter, multilinear and alternating forms, the wedge product, exterior derivative, and pullback are introduced, then applied to line and surface integrals, culminating in the Stokes, Green, and Gauss-Ostrogradsky theorems, with worked exercises. Draws on Bahayou [2], do Carmo [4], Hélein-Chaudouard [5], Tu [7], and Waldspurger [9].

3.1 Multilinear Forms on $\mathbb{R}^n$

3.1.1 Alternating Multilinear Forms on $\mathbb{R}^n$

Definition 3.1. A function $f : \mathbb{R}^n \times ... \times \mathbb{R}^n = (\mathbb{R}^n)^p \longrightarrow \mathbb{R}$, $p \in \mathbb{N}^*$ is called a $p-$linear or multilinear form of order p (and a bilinear form when $p = 2$) if it is linear in each of its arguments. That is, for all $u_i, v_i \in \mathbb{R}^n$, $i = 1, ..., p$ and λ, $\mu \in \mathbb{R}$, we have

$$f(u_1..., \lambda u_i + \mu v_i, ..., u_p) = \lambda f(u_1..., u_i, ..., u_p) + \mu f(u_1..., v_i, ..., u_p).$$

Definition 3.2. A $p-$linear form f is called alternating if it changes sign whenever any two of its arguments are swapped. More precisely, for all $v_1, ..., v_p \in \mathbb{R}^n$ and $1 \leq i < j \leq p$, we

have

$$f(v_1, ..., v_i, ..., v_j, ..., v_p) = -f(v_1..., v_j, ..., v_i, ..., v_p).$$

Notations. We denote by $L_p(\mathbb{R}^n, \mathbb{R})$ the space of $p-$linear forms, and by $A^p(\mathbb{R}^n, \mathbb{R})$ the space of alternating $p-$linear forms.

Example 3.1. One of the most well-known examples of alternating bilinear forms is the determinant of 2×2 square matrices. Define

$$\begin{array}{rccl} f: & \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ & (u, v) & \mapsto & f(u, v) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \end{array},$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. This form is bilinear (linear in each argument) and alternating, satisfying the antisymmetry property: $f(v, u) = -f(u, v)$.

Definition 3.3. A permutation of the set $\{1, 2, ..., p\}$ is any bijective function from $\{1, 2, ..., p\}$ onto itself. We denote by $S(p)$ to the set of all permutations of $\{1, 2, ..., p\}$.

Definition 3.4. We define the map ε by

$$\begin{array}{rccl} \varepsilon: & S(p) & \to & \{-1, +1\} \\ & \sigma & \mapsto & \varepsilon(\sigma) = \begin{cases} +1 \text{ if } \sigma \text{ is an even permutation,} \\ -1 \text{ if } \sigma \text{ is an odd permutation,} \end{cases} \end{array}.$$

The value $\varepsilon(\sigma)$ is called the signature of the permutation σ.

Proposition 3.1. Let $f \in L_p(\mathbb{R}^n, \mathbb{R})$ and $\sigma \in S(p)$. We define the map (σf) by

$$\begin{array}{rccl} (\sigma f): & \mathbb{R}^n \times ... \times \mathbb{R}^n = (\mathbb{R}^n)^p & \to & \mathbb{R} \\ & (x_1, x_2, ..., x_p) & \mapsto & f\left(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(p)}\right) \end{array}.$$

This map satisfies the following properties:

1. $(\sigma f) = \varepsilon(\sigma) f$, i.e.,

$$f\left(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(p)}\right) = \varepsilon(\sigma) f(x_1, x_2, ..., x_p).$$

2. For any $\sigma_1, \sigma_2 \in S(p)$,

$$\varepsilon(\sigma_1\sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2).$$

3.1.2 Exterior Product of Multilinear Forms

Definition 3.5. Let $f \in A^p(\mathbb{R}^n, \mathbb{R})$ and $g \in A^q(\mathbb{R}^n, \mathbb{R})$. We define the map $f \wedge g$ by

$$\begin{array}{rccl} f \wedge g: & (\mathbb{R}^n)^{p+q} & \to & \mathbb{R} \\ & (x_1, ..., x_p, x_{p+1}, ..., x_{p+q}) & \mapsto & f \wedge g(x_1, ..., x_{p+q}) \end{array}$$

where

$$f \wedge g(x_1, ..., x_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S(p+q)} \varepsilon(\sigma) . f\left(x_{\sigma(1)}, ..., x_{\sigma(p)}\right) . g\left(x_{\sigma(p+1)}, ..., x_{\sigma(p+q)}\right)$$

Definition 3.6. The map $f \wedge g$ defined above is called the exterior product (or wedge product) of f and g, and is denoted by $f \wedge g$.

Definition 3.7. Let $f_1, ..., f_p$ be p linear forms on $\mathbb{R}^n$. The exterior product of the p linear forms (f_i), for $i \in \{1, 2, ..., p\}$, is the map defined by:

$$\begin{array}{rccl} f_1 \wedge ... \wedge f_p: & (\mathbb{R}^n)^p & \to & \mathbb{R} \\ & (x_1, ..., x_p) & \mapsto & f_1 \wedge ... \wedge f_p(x_1, ..., x_p) \end{array},$$

where

$$f_1 \wedge ... \wedge f_p(x_1, ..., x_p) = \det(f_i(x_j)), \quad 1 \leq i, j \leq p.$$

Corollary 3.1. Let f_p be an alternating $p-$linear form on $\mathbb{R}^p$. Then for all $x_1, ..., x_p \in \mathbb{R}^p$, we have

$$f_p(x_1, ..., x_p) = f_p(e_1, ..., e_p) \det[x_{ij}], \quad 1 \leq i, j \leq p.$$

Proof. Let $\{e_1, e_2\}$ be a basis of $\mathbb{R}^2$, and let $x, y \in \mathbb{R}^2$, we have

$$x = x_1e_1 + x_2e_2, \ y = y_1e_1 + y_2e_2.$$

Then, for an alternating bilinear form f, we have

$$\begin{aligned} f(x,y) &= f(x_1e_1 + x_2e_2, y_1e_1 + y_2e_2) = x_1 f(e_1, y_1e_1 + y_2e_2) + x_2 f(e_2, y_1e_1 + y_2e_2) \\ &= x_1y_1 f(e_1,e_1) + x_1y_2 f(e_1,e_2) + y_1x_2 f(e_2,e_1) + x_2y_2 f(e_2,e_2) \\ &= x_1y_2 f(e_1,e_2) + y_1x_2 f(e_2,e_1) \\ &= x_1y_2 f(e_1,e_2) - y_1x_2 f(e_1,e_2) \ \ (\text{since} \ \ f(e_2,e_1) = -f(e_1,e_2)) \\ &= \det(x,y)\, f(e_1,e_2) \end{aligned}$$

where we have used $f(e_1,e_1) = f(e_2,e_2) = 0$ due to the alternating property of f.

Definition 3.8. If $\{e_1, ..., e_n\}$ is a basis of $\mathbb{R}^n$, the dual basis $\{e_1^*, ..., e_n^*\}$ is defined by:

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } i, j = 1, ..., n.$$

3.2 Differential Forms on an Open Subset U of $\mathbb{R}^n$

In this section, we introduce the notion of differential forms defined on open subsets of $\mathbb{R}^n$. These forms generalize functions and vector fields and provide a natural framework for integration and the formulation of key results in calculus and geometry, such as Stokes' theorem.

Definition 3.9. Let U be an open set of $\mathbb{R}^n$, and let p, k be natural numbers. A differential form of degree p and of class C^k on U is any function ω from U to the set of alternating $p-$linear forms:

$$\begin{aligned} \omega : \ U &\to A^p(\mathbb{R}^n, \mathbb{R}) \\ x &\mapsto \omega(x) = \sum_{1 \leq i_1 < ... < i_p \leq n} f_{i_1,...,i_p}(x)\, dx_{i_1} \wedge ... \wedge dx_{i_p}, \end{aligned}$$

where the functions $f_{i_1,...,i_p}$ are of class class C^k on U.

The set of all differential forms of degree p and class C^k on U is denoted by: $\Omega^p_{C^k}(U, \mathbb{R})$.

Example 3.2. We present some basic examples of differential forms:

1. Any function f defined on an open set $U \subset \mathbb{R}^n$ is a differential form of degree 0.

2. A differential form ω of degree 1 on an open set $U \subset \mathbb{R}^3$ is given by:

$$\omega = Pdx + Qdy + Rdz$$

where P, Q and R are functions defined on the open set U.

3. If $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function of class C^k, then the map

$$x \longmapsto df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i$$

defines a differential form of degree 1 on U.

Remark 3.1. Let U be an open subset of $\mathbb{R}^n$, and consider a differential form of degree p on U.

If $p > n$, then this form is necessarily zero at every point in U.

3.2.1 Exterior Product of Differential Forms

Definition 3.10. Let ω and α be differential forms on an open set $U \subset \mathbb{R}^n$ of degrees p and r, respectively.The wedge product of alternating multilinear forms extends naturally to differential forms. That is, if

$$\omega(x) = \sum_{1 \le i_1 < \ldots < i_p \le n} f_{i_1,\ldots,i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \text{ et } \alpha(x) = \sum_{1 \le i_1 < \ldots < i_r \le n} g_{i_1,\ldots,i_r}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_r},$$

then the wedge product $\omega \wedge \alpha$ is given by:

$$(\omega \wedge \alpha)(x) = \omega(x) \wedge \alpha(x) = \sum_{\substack{1 \le i_1 < \ldots < i_p \le n \\ 1 \le j_1 < \ldots < j_r \le n}} f_{i_1,\ldots,i_p}(x)\, g_{j_1,\ldots,j_r}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_r}.$$

Example 3.3. Let $f, g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be two functions of class C^1 on U. Then the wedge product of their differential forms of degree 1:

$$df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i \quad \text{and} \quad dg(x) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x)\, dx_i,$$

is the differential form of degree 2, given by:

$$df(x) \wedge dg(x) = \sum_{1 \leq i < j \leq n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)(x)\, dx_i \wedge dx_j.$$

In particular, when $n = 3$, we have

$$(df \wedge dg) = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) dx \wedge dy + \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \right) dx \wedge dz + \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) dy \wedge dz.$$

From this formula, it follows that $df \wedge df = 0$, and in particular,

$$dx \wedge dx = 0,\ dy \wedge dy = 0 \ \text{ and } \ dz \wedge dz = 0.$$

3.2.2 Exterior Derivative of a Differential Form

Definition 3.11. Let U be an open subset of $\mathbb{R}^n$, and let p, k be positive integers. Let ω be a differential form of degree p and class C^k on U, given by:

$$\omega(x) = \sum_{1 \leq i_1 < \ldots < i_p \leq n} f_{i_1, \ldots, i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

The exterior derivative of ω is the differential form $d\omega$ of degree $p+1$ defined by:

$$d\omega(x) = \sum_{1 \leq i_1 < \ldots < i_p \leq n,\ 1 \leq j \leq n} \left(\frac{\partial f_{i_1, \ldots, i_p}(x)}{\partial x_j} \right) dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

This defines a linear map d from $\Omega^p_{C^k}(U, \mathbb{R})$ to $\Omega^{p+1}_{C^{k-1}}(U, \mathbb{R})$, which sends differential forms of degree p to differential forms of degree $p+1$.

Example 3.4. We consider the following examples:

1. Let $f(x, y, z) = x^2 y + \sin z$, which is a differential form of degree 0. Its exterior derivative is simply the classical differential:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 2xy dx + x^2 dy + \cos z dz$$

   Thus, df is a differential form of degree 1.

2. Let $\omega = Pdx+Qdy+Rdz$ be a differential form of degree 1. Then its exterior derivative is

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

We see that $d\omega$ is a differential form of degree 2.

Here are the fundamental properties of the exterior derivative.

Proposition 3.2. Let ω and α be differential forms of degrees p and r , respectively, and of class C^k on an open set $U \subset \mathbb{R}^n$. Thent he exterior derivative d satisfies the following properties:

1. Linearity of d:

$$d(\omega + \alpha) = d\omega + d\alpha.$$

The operator d is linear, just like the classical derivative.

2. Leibniz Rule (for the wedge product):

$$d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha.$$

This is a generalization of the Leibniz rule to the wedge product. The sign $(-1)^p$ appears because the operator d must "pass over" the form ω (of degree p) to act on α. This involves p permutations, introducing a minus sign if p is odd. If $p = 0$, we recover:

$$d(\omega\alpha) = d\omega \wedge \alpha + \omega d\alpha$$

3. Nilpotency:

$$d(d\omega) = 0.$$

The exterior derivative applied twice always yields zero.

Definition 3.12. Let ω be a differential form of degree p and class C^k on an open set $U \subset \mathbb{R}^n$.

- ω is said to be closed if $d\omega = 0$.
- ω is said to be exact if there exists a differential form α of degree $(p-1)$ such that $d\alpha = \omega$.

Remark 3.2. Every exact form is automatically closed, since $d(d\omega) = 0$. However, the converse is not always true: a closed form is not necessarily exact. Whether a closed form is also exact depends on the topology of the domain under consideration.

Definition 3.13. We introduce the following concepts:

1. Let a, $b \in \mathbb{R}^n$. A path in $\mathbb{R}^n$ with endpoints a and b is any continuous map $f : [t_0, t_1] \longrightarrow \mathbb{R}^n$ such that $f(t_0) = a$ and $f(t_1) = b$.
2. If the initial and final points coincide, i.e., $f(t_0) = f(t_1)$, the path is called closed or a loop.
3. A path is said to be simple if it does not intersect itself, i.e., it has no double points.
4. A subset $E \subset \mathbb{R}^n$ is said to be path-connected if, for every pair of points a, $b \in E$, there exists a path contained entirely in E joining a and b.

Definition 3.14. A set $E \subset \mathbb{R}^n$ is said to be simply connected if it satisfies the following two conditions:

1. It is path-connected, meaning that any two points in E can be joined by a continuous path lying entirely within E.
2. Every simple closed path in E can be continuously contracted to a point within E.

Definition 3.15. An open set $U \subset \mathbb{R}^n$ is called star-shaped if there exists a point $a \in U$ such that, for every $x \in U$, the line segment $[a, x] = \{\lambda x + (1-\lambda)\, a \ / \ 0 \leq \lambda \leq 1\}$ is entirely contained in U.

Lemma 3.1. (Poincaré lemma) Let $U \subset \mathbb{R}^n$ be a star-shaped open set. Then every closed differential form defined on U is exact.

Remark 3.3. The property of being star-shaped is a stronger geometric condition than simple connectedness. Every star-shaped domain is simply connected, but the converse is not necessarily true.

Example 3.5. Consider the differential form on $\mathbb{R}^3$ defined by:

$$\omega(x, y, z) = (-2y + yz)\, dx \wedge dy + 2x dx \wedge dz - xy dy \wedge dz.$$

Its exterior derivative is given by:

$$d\omega(x, y, z) = (y - 0 - y)\, dx \wedge dy \wedge dz = 0.$$

Thus, ω is closed. According to the Poincaré Lemma, since $\mathbb{R}^3$ is star-shaped, there exists a differential form α such that $d\alpha = \omega$. We now verify that the differential form α of degree 1 defined by:

$$\alpha(x, y, z) = \left(y^2 - 2xz\right) dx + xyz dy,$$

satisfies $d\alpha = \omega$, confirming that α is indeed a primitive form (or potential form) of ω.

3.2.3 Pullback of Differential Forms – Change of Variables

Definition 3.16. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and let $\phi : V \longrightarrow U$ be a C^k map. Let ω be a differential U of degree p, expressed as

$$\omega(x) = \sum_{1 \leq i_1 < \ldots < i_p \leq n} f_{i_1, \ldots, i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

The pullback of ω by ϕ, denoted $\phi^* \omega$, is the differential form of degree p on V defined by

$$\phi^* \omega(u) = \sum_{1 \leq i_1 < \ldots < i_p \leq n} f_{i_1, \ldots, i_p}(\phi(u))\, d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p}.$$

where $d\phi_i = d(\phi_i)$ denotes the exterior derivative of the i-th component function of ϕ.

This operation allows one to transfer, or "pull back" a differential form defined on U to a form defined on V, preserving its degree and differentiable structure, while adapting it to the new coordinate system determined by ϕ.

Example 3.6. Let ω be a differential form of degree 2 on $\mathbb{R}^3$, given by:

$$\omega(x, y, z) = dx \wedge dy$$

We define a change of variables $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by:

$$\phi(r, u, v) = (x, y, z) = (r \cos u \cos v, r \sin u \cos v, r \sin v).$$

The pullback of ω by ϕ is

$$\begin{aligned}
\phi^*\omega(r, u, v) &= d(r \cos u \cos v) \wedge d(r \sin u \cos v) \\
&= (\cos u \cos v dr - r \sin u \cos v du - r \cos u \sin v dv) \wedge \\
&\qquad (\sin u \cos v dr + r \cos u \cos v du - r \sin u \sin v dv) \\
&= r \cos^2 v dr \wedge du + r^2 \sin v \cos v du \wedge dv.
\end{aligned}$$

Thus, the pullback $\phi^*\omega$ is expressed in the (r, u, v) coordinates as

$$\phi^*\omega(r, u, v) = r \cos^2 v dr \wedge du + r^2 \sin v \cos v du \wedge dv.$$

Properties 3.1. The fundamental properties of the pullback are:

1. Linearity of the pullback:

$$\phi^*(\omega + \alpha) = \phi^*\omega + \phi^*\alpha.$$

2. Compatibility with the wedge product:

$$\phi^*(\omega \wedge \alpha) = \phi^*\omega \wedge \phi^*\alpha.$$

3. Commutation with the exterior derivative:

$$d(\phi^*\omega) = \phi^* d(\omega).$$

4. Pullback of a composition of maps:

$$(\phi \circ \theta)^* \omega = \phi^* (\theta^* \omega).$$

These properties are powerful tools for working with differential forms and geometric transformations on manifolds. They allow one to manipulate and transform forms effectively, while preserving the fundamental geometric relationships.

3.2.4 The Operators: Gradient, Divergence, and Curl

Definition 3.17. Let $U \subset \mathbb{R}^n$. A vector field on U is a function $X : U \longrightarrow \mathbb{R}^n$ that assigns to each point $p \in U$ a vector $X(p) \in \mathbb{R}^n$.

Definition 3.18. Let f be a continuously differentiable function (i.e., of class C^1) defined on an open subset $U \subset \mathbb{R}^n$. The gradient of f , denoted $\nabla f(x)$ or $\operatorname{grad} f(x)$, is the vector field defined by:

$$\operatorname{grad} f(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x) \right), \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

Definition 3.19. Let V be a vector field of class C^1 on an open subset of $\mathbb{R}^3$, given by:

$$V(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

The divergence of V, denoted $\nabla . V$ or $\operatorname{div} V$, is defined by:

$$\operatorname{div} V = \nabla . V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) . (P, Q, R) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The curl of V, denoted $\nabla \wedge V$ or $\operatorname{curl} V$, is defined by:

$$\operatorname{curl} V = \nabla \wedge V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \wedge (P, Q, R) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark 3.4. In terms of differential forms, if $V = (P, Q, R)$, $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ and $\alpha = Pdx + Qdy + Rdz$, then

$$d\omega = (\operatorname{div} V)\, dx \wedge dy \wedge dz,$$

$$d\alpha = (\operatorname{curl} V)_1\, dy \wedge dz + (\operatorname{curl} V)_2\, dz \wedge dx + (\operatorname{curl} V)_3\, dx \wedge dy.$$

Definition 3.20. Let V be a vector field on an open subset of $\mathbb{R}^n$. If $V = \operatorname{grad} f$, then V is said to be derived from the potential f, or simply a gradient field.

Properties 3.2. The following identities hold:

1. If $f \in C^2(\mathbb{R}^3)$, then $\operatorname{curl}(\operatorname{grad} f) = 0$.
2. If $V \in C^2(\mathbb{R}^3)$, then $\operatorname{div}(\operatorname{curl} V) = 0$.

3.3 Integration of Differential Forms

Let ω be a differential form of degree p and of class C^k on an open set $U \subset \mathbb{R}^n$, expressed as:

$$\omega(x) = \sum_{1 \le i_1 < \ldots < i_p \le n} f_{i_1,\ldots,i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}, \quad x \in U.$$

Let $\phi : D \subset \mathbb{R}^m \longrightarrow U \subset \mathbb{R}^n$ be a map of class C^k. The pullback of ω, denoted $\phi^*\omega$, is a differential form of degree p on $D \subset \mathbb{R}^m$ given by:

$$\phi^*\omega(u) = \sum_{1 \le i_1 < \ldots < i_p \le n} f_{i_1,\ldots,i_p}(\phi(u))\, d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p}.$$

where $d\phi_i = d(\phi_i)$ denotes the exterior derivative of the i-th component function of ϕ**.**

In particular, if $p = m$, then $\phi^*\omega(u)$ can be written as

$$\phi^*\omega(u) = h(u)\, du = h(u)\, du_1 \wedge \ldots \wedge du_m$$

where h is a real-valued function on $D \subset \mathbb{R}^m$.

Definition 3.21. The differential form ω of degree p is said to be integrable over the oriented submanifold U, of dimesion p, if the function h is integrable on D. In this case, the integral of ω over U is defined by:

$$\int_U \omega = \int_D h(u)\, du$$

where the right-hand side is a multiple integral.

When $p = 1$, the integral is called a line integral, and when $p = 2$, it is referred to as a surface integral. We will focus here on these two particular types of integrals.

Before proceeding with their definitions, we briefly recall some basics of parameterized curves and surfaces.

3.3.1 Parameterized Curves and Surfaces

In this section, we introduce some foundational concepts related to regular parameterized curves and surfaces.

Definition 3.22. A path or arc of class C^k in $\mathbb{R}^n$ is a function of class C^k from a real interval $I \subset \mathbb{R}$, into $\mathbb{R}^n$ with $n \geq 2$, defined by:

$$\begin{array}{rccl} \phi: & I \subset \mathbb{R} & \to & \mathbb{R}^n \\ & t & \mapsto & \phi(t) = (\phi_1(t), ..., \phi_n(t)). \end{array}$$

Remark 3.5. We have the following remarks:

1. If $I = [a, b]$ with $a < b$, the points $A = \phi(a)$ and $B = \phi(b)$ are called the initial and final points of ϕ, respectively.
2. If $\phi(a) = \phi(b)$, the path is said to be closed or to form a loop.
3. A path ϕ is often denoted by the pair (I, ϕ).

Definition 3.23. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}^n$ be a C^kfunction. The set

$$\gamma = \phi(I) = \{\phi(t) \in \mathbb{R}^n,\ t \in I\},$$

is called the support of ϕ, or a curve in $\mathbb{R}^n$ parametrized by the function ϕ.

By convention, the path ϕ is often identified with its image γ, and is simply referred to as a parametrized path of class C^k.

Example 3.7. We consider the following examples:

1. The function $\phi(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$, defines a closed path in $\mathbb{R}^2$.

2. The segment with endpoints A and B, denoted $[AB]$, is a curve parameterized by the function $\phi(t) = (1-t)A + tB$, $0 \leq t \leq 1$.

Definition 3.24. A parametrized surface of class C^k is a function ϕ of class C^k from a connected open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^n$, with $n \geq 3$.

$$\begin{array}{rccl} \phi: & U \subset \mathbb{R}^2 & \rightarrow & \mathbb{R}^n \\ & (u,v) & \mapsto & \phi(u,v) = (\phi_1(u,v), ..., \phi_n(u,v)). \end{array}$$

The image $S = \phi(U) = \{\phi(u,v) \in \mathbb{R}^n, \ (u,v) \in U\}$ is called the geometric support of the parametrized surface ϕ.

- If, at some $(u_0, v_0) \in U$, the vectors $\frac{\partial \phi(u,v)}{\partial u}$ and $\frac{\partial \phi(u,v)}{\partial v}$ are linearly independent, the plane they span is the tangent plane to the surface at the point (u_0, v_0).

- Their cross product $\eta(u_0, v_0) = \frac{\partial \phi}{\partial u}(u_0, v_0) \times \frac{\partial \phi}{\partial v}(u_0, v_0)$ is a normal vector to the surface (orthogonal to the tangent plane).

- The unit normal vector $n(u_0, v_0) = \frac{\eta(u_0,v_0)}{\|\eta(u_0,v_0)\|}$ is called the oriented normal at the point (u_0, v_0).

3.3.2 Line Integral of a Function

The concept of a line integral can be introduced either for a scalar function $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ or, more generally, for a differential form ω of degree 1.

Definition 3.25. Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. The line integral of f along the curve γ parameterized by $\phi : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ is defined by

$$\int_\gamma f = \int_a^b f(\phi(t)) \|\phi'(t)\| \, dt,$$

where

$$\|\phi'(t)\| = \sqrt{\sum_{i=1}^{n} (\phi_i'(t))^2} = \sqrt{(\phi_1'(t))^2 + ... + (\phi_n'(t))^2}.$$

Remarks 3.6. We have the following remarks:

1. The definition of the line integral does not depend on the choice of parameterization of the curve γ.

2. In particular, if $f = 1$ on γ, the line integral reduces to the length of the curve:

$$|\gamma| = \int_a^b \|\phi'(t)\| \, dt.$$

This formula computes the total arc length of the curve γ.

Example 3.8. Let us compute the length of the curve

$$C = \left\{(x, y) \in \mathbb{R}^2 : y = x^2 \text{ and } -1 \leq x \leq 1\right\}.$$

This curve can be parameterized by

$$C = \left\{(x(t), y(t)) = \left(t, t^2\right) \in \mathbb{R}^2, \ -1 \leq t \leq 1\right\}.$$

The length of C is given by the line integral:

$$|C| = \int_{-1}^{1} \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_{-1}^{1} \sqrt{1 + 4t^2} dt = \sqrt{5} + \frac{1}{4} \ln\left(\sqrt{5} + 2\right).$$

3.3.3 Line Integral of a Differential Form of Degree 1

Definition 3.26. Let ω be a continuous differential form of degree 1 on an open set $U \subset \mathbb{R}^n$ containing the curve $\phi([a, b])$. Write

$$\omega = \sum_{i=1}^{n} \omega_i(x) \, dx_i.$$

The line integral of the differential form ω along the path ϕ is defined by

$$\int_\gamma \omega = \int_a^b \sum_{i=1}^{n} \omega_i(\phi(t)) \, \phi_i'(t) \, dt.$$

This integral represents the total work done by the force field $\omega = \sum_{i=1}^{n} \omega_i(x)\, e_i$, when a particle moves along the trajectory γ.

Examples 3.9. The following examples illustrate the computation of line integrals and work along curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

1. Calculate the integral $\int_\gamma x^2 dx - xydy$, where γ is the semicircular arc centered at the origin O, with radius 1, starting at $A(1,0)$ and ending at $B(-1,0)$. The curve γ can be parameterized by

$$\begin{array}{rccl} \phi: & [0,\pi] & \to & \mathbb{R}^2 \\ & t & \mapsto & (x(t), y(t)) = (\cos t, \sin t). \end{array}$$

   Then,

$$\begin{aligned} \int_\gamma x^2 dx - xydy &= \int_0^\pi \left(\cos^2 t\,(-\sin t) - \cos t \sin t\,(\cos t)\right) dt \\ &= -2\int_0^\pi \cos^2 t \sin t dt = \left[\frac{2}{3}\cos^3 t\right]_0^\pi = -\frac{4}{3}. \end{aligned}$$

2. Calculate the work $W_{0;3\pi}$ done by the force field F defined on $\mathbb{R}^3$ by: $F(x,y,z) = (4y, 0, 2z)$, along the curve γ parameterized by:

$$\begin{array}{rccl} \phi: & [0,3\pi] & \to & \mathbb{R}^3 \\ & t & \mapsto & (x(t), y(t), z(t)) = (\cos t, \sin t, t). \end{array}$$

   We have

$$\begin{aligned} W_{0;3\pi} &= \int_0^{3\pi} (4\sin t, 0, 2t).(-\sin t, \cos t, 1)\, dt \\ &= \int_0^{3\pi} \left(-4\sin^2 t + 2t\right) dt = \left[-2t + t^2 + \sin 2t\right]_0^{3\pi} = 9\pi^2 - 6\pi. \end{aligned}$$

3.3.4 Surface Integral of a Function

One can define the surface integral for functions defined on an open set $U \subset \mathbb{R}^n$ and for differential forms of degree 2 on U.

Let D be a connected open set in $\mathbb{R}^2$ and S the surface parametrized in $\mathbb{R}^n$ by the map

$$\phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n, \quad n \geq 3.$$

Definition 3.27. Let f be a continuous function defined on an open set $U \subset \mathbb{R}^n$ that contains the surface $S = \phi(D)$. The surface integral of f over S is defined by

$$\int_S f = \int\int_D f(u,v) \left\| \frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) \right\| dudv.$$

Remark 3.7. If $f = 1$, we obtain

$$Aire(S) = \int_S f = \int\int_D \left\| \frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) \right\| dudv.$$

Example 3.10. Let us compute the surface area of the sphere

$$\mathbb{S}^2 = \left\{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2 \right\}.$$

The sphere $\mathbb{S}^2$ can be parametrized by the function

$$\begin{array}{rlcl} \phi: & D = [0, 2\pi[\times [0, \pi[& \to & \mathbb{R}^3 \\ & (u,v) & \mapsto & (x,y,z)(u,v) = (r\cos u \sin v, r \sin u \sin v, r \cos v). \end{array}$$

Let us compute the cross product of the partial derivatives

$$\frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) = r^2 \left(-\cos u \sin^2 v, -\sin u \sin^2 v, -\cos v \sin v \right).$$

Taking the norm, we obtain

$$\begin{aligned} \left\| \frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) \right\| &= r^2 \sqrt{\cos^2 u \sin^4 v + \sin^2 u \sin^4 v + \cos^2 u \sin^2 v} \\ &= r^2 \sqrt{\sin^4 v + \cos^2 u \sin^2 v} = r^2 \sin v. \end{aligned}$$

Thus, the surface area of the sphere is

$$Aire\left(\mathbb{S}^2\right) = \int_S f = \int\int_D \left\| \frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) \right\| dudv = \int_0^{2\pi} \int_0^{\pi} r^2 \sin v du dv = 4\pi r^2.$$

Definition 3.28. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension d (with $n \geq 2$), and let $f : M \longrightarrow \mathbb{R}$, be a continuous function. Suppose $\phi : U \longrightarrow M$, is a smooth

parametrization of M , where U is an open subset of $\mathbb{R}^d$. Then the integral of f over the submanifold M is defined by:

$$\int_M f = \int_U f(\phi(u)) \sqrt{\det\left((D\phi)^T D\phi\right)} du = \int_U f(\phi(u)) \sqrt{\det(g_{ij})} du_1 ... du_d$$

where:

- $u = (u_1, ..., u_d) \in \mathbb{R}^d$.
- $D\phi(u) \in \mathbb{R}^{n \times d}$ is the Jacobian matrix of ϕ, with entries:

$$(D\phi(u))_{ij} = \frac{\partial \phi_i}{\partial u_j}(u), \ i \leq 1 \leq n, \ 1 \leq j \leq d.$$

- $(g_{ij}(u))_{1 \leq i,j \leq d}$ is the induced metric (also known as the first fundamental form), defined by:

$$g_{ij}(u) = \left\langle \frac{\partial \phi}{\partial u_i}(u), \frac{\partial \phi}{\partial u_j}(u) \right\rangle, \ 1 \leq i, j \leq d.$$

Example 3.11. Let us return to the previous example where $M = \mathbb{S}^2$, $n = 3$, $d = 2$, and ϕ is the spherical parametrization given by:

$$\begin{array}{rccl} \phi: & U = [0, 2\pi[\times [0, \pi[& \rightarrow & \mathbb{R}^3 \\ & (u, v) & \mapsto & (x, y, z)(u, v) = (r\cos u \sin v, r \sin u \sin v, r \cos v). \end{array}$$

The Jacobian matrix of ϕ is

$$D\phi = \begin{pmatrix} -r\sin u \sin v & r\cos u \cos v \\ r\cos u \sin v & r \sin u \cos v \\ 0 & -r \sin v \end{pmatrix}, \ (D\phi)^T = \begin{pmatrix} -r\sin u \sin v & r\cos u \sin v & 0 \\ r\cos u \cos v & r \sin u \cos v & -r \sin v \end{pmatrix}.$$

Then the product $(D\phi)^T D\phi$, gives the metric tensor

$$(D\phi)^T D\phi = \begin{pmatrix} r^2 \sin^2 v & 0 \\ 0 & r^2 \end{pmatrix}.$$

Hence, the area of the sphere is

$$Aire\left(\mathbb{S}^2\right) = \int_U \sqrt{\det\left((D\phi)^T D\phi\right)} du = \int_U \sqrt{\det(g_{ij})} du dv = 4\pi r^2$$

3.3.5 Surface Integral of a Differential Form of Degree 2

Let ω a differential form of degree 2 and of class C^k defined on an open set $U \subset \mathbb{R}^n$ containing the surface $S = \phi(D)$, given by:

$$\omega(x) = \sum_{i,j=1}^{n} f_{i,j}(x)\, dx_i \wedge dx_j, \quad x = (x_1, ..., x_n).$$

The pullback of ω is a differential form of degree 2 defined on $D \subset \mathbb{R}^2$, given by:

$$\phi^*\omega(u,v) = \sum_{i,j=1}^{n} f_{i,j}(\phi(u,v))\, d\phi_i \wedge d\phi_j.$$

Since

$$\begin{aligned} d\phi_i \wedge d\phi_j &= \left(\frac{\partial \phi_i(u,v)}{\partial u}du + \frac{\partial \phi_i(u,v)}{\partial v}dv\right) \wedge \left(\frac{\partial \phi_j(u,v)}{\partial u}du + \frac{\partial \phi_j(u,v)}{\partial v}dv\right) \\ &= \frac{D(\phi_i, \phi_j)}{D(u,v)}(u,v)\, du \wedge dv; \end{aligned}$$

where

$$\frac{D(\phi_i, \phi_j)}{D(u,v)}(u,v) = \det J(\phi_i, \phi_j) = \det \begin{pmatrix} \frac{\partial \phi_i}{\partial u} & \frac{\partial \phi_i}{\partial v} \\ \frac{\partial \phi_j}{\partial u} & \frac{\partial \phi_j}{\partial v} \end{pmatrix} = \left(\frac{\partial \phi_i}{\partial u}\frac{\partial \phi_j}{\partial v} - \frac{\partial \phi_j}{\partial u}\frac{\partial \phi_i}{\partial v}\right),$$

then

$$\phi^*\omega(u,v) = \left(\sum_{i=1}^{n}\sum_{j=i+1}^{n} f_{i,j}(\phi(u,v))\frac{D(\phi_i, \phi_j)}{D(u,v)}(u,v)\right) du \wedge dv.$$

Definition 3.29. With the previous notations, the integral of the differential form ω over the surface S is defined by:

$$\int_S \omega = \int\int_D \left(\sum_{i,j=1}^{n} f_{i,j}(\phi(u,v))\frac{D(\phi_i, \phi_j)}{D(u,v)}(u,v)\right) dudv.$$

In particular, if ω is a differential form of degree 2 defined on an open subset $U \subset \mathbb{R}^3$, i.e

$$\omega(x,y,z) = P(x,y,z)\, dy \wedge dz + Q(x,y,z)\, dz \wedge dx + R(x,y,z)\, dx \wedge dy,$$

and if $\phi(u,v) = (\phi_1(u,v), \phi_2(u,v), \phi_3(u,v))$, then

$$\int_S \omega = \int\int_D \left(P(\phi(u,v))\frac{D(\phi_2, \phi_3)}{D(u,v)} + Q(\phi(u,v))\frac{D(\phi_3, \phi_1)}{D(u,v)} + R(\phi(u,v))\frac{D(\phi_1, \phi_2)}{D(u,v)}\right) dudv.$$

If we set $V = (P, Q, R)$, then

$$\int_S \omega = \int\int_D V(\phi(u,v)).N(u,v)\,dudv,$$

where $N = \left(\frac{D(\phi_2,\phi_3)}{D(u,v)}, \frac{D(\phi_3,\phi_1)}{D(u,v)}, \frac{D(\phi_1,\phi_2)}{D(u,v)}\right) = \frac{\partial\phi(u,v)}{\partial u} \wedge \frac{\partial\phi(u,v)}{\partial v}$ is the normal vector to the surface S.

Example 3.12. Let $S^+ = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 4,\ z \geq 0\}$ be the upper hemisphere of radius 2 centered at the origin. We want to calculate the surface integral:

$$I = \int_{S^+} xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

The upper hemisphere S^+ can be parametrized by the map

$$\begin{array}{rlcl} \phi: & D = [0, 2\pi[\times \left[0, \frac{\pi}{2}\right] & \to & \mathbb{R}^3 \\ & (u,v) & \mapsto & (x,y,z)(u,v) = (2\cos u \sin v, 2\sin u \sin v, 2\cos v). \end{array}$$

We have

$$\begin{aligned} dy \wedge dz &= \det \begin{pmatrix} 2\cos u \sin v & 2\sin u \cos v \\ 0 & -2\sin v \end{pmatrix} du \wedge dv \\ &= -4\cos u \sin^2 v du \wedge dv. \end{aligned}$$

Similarly, we obtain

$$dz \wedge dx = -4\sin u \sin^2 v du \wedge dv \text{ and } dx \wedge dy = -4\cos v \sin v du \wedge dv.$$

Therefore, the integral becomes

$$\begin{aligned} I &= 8\int\int_D \begin{bmatrix} (\cos u \sin v)\left(-\cos u \sin^2 v\right) \\ +(\sin u \sin v)\left(-\sin u \sin^2 v\right) + (\cos v)\left(-\cos v \sin v\right) \end{bmatrix} du \wedge dv \\ &= -8\int\int_D \left(\cos^2 u \sin^3 v + \sin^2 u \sin^3 v + \cos^2 v \sin v\right) du \wedge dv \\ &= -8\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\sin^3 v + \sin^3 v + \cos^2 v \sin v\right) dudv \\ &= -8\int_0^{2\pi} du \int_0^{\frac{\pi}{2}} \sin v dv = -16\pi. \end{aligned}$$

3.4 Stokes', Green's, and Gauss-Ostrogradsky's Theorems

The Fundamental Theorem of Calculus states that the integral of a function f of one real variable over an interval $[a, b] \subset \mathbb{R}$ is equal to the change of an antiderivative F of f on the boundary of the interval, i.e.,

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{with} \quad F'(x) = f(x).$$

This is actually the one-dimensional case of Stokes' Theorem, which relates the integral of the exterior derivative $d\omega$ of a differential form ω over an oriented submanifold $M \subset \mathbb{R}^n$, to the integral of ω over the boundary ∂M of that manifold, that is:

$$\int_M d\omega = \int_{\partial M} \omega,$$

The classical Green-Riemann Theorem (in dimension 2) and the Gauss-Ostrogradsky Theorem (also known as the Divergence Theorem, in dimension 3) are particular cases of this general result.

3.4.1 Green-Riemann Formula in $\mathbb{R}^2$

The Green-Riemann formula establishes a relationship between the integral of a differential form ω of degree 1 over a simple closed curve γ in $\mathbb{R}^2$, and a double integral over the domain enclosed by γ. As mentioned above, this formula can be seen as an analogue of the Fundamental Theorem of Calculus for functions of one variable.

Theorem 3.1. Let $\omega = Pdx + Qdy$ be a differential form of degree 1 and of class C^1 defined on an open set $U \subset \mathbb{R}^2$ containing a compact domain K with a boundary of class C^1 (i.e., a curve of class C^1). If ∂K denotes the oriented boundary of K, then

$$\int_{\partial K} P(x,y)\,dx + Q(x,y)\,dy = \int\int_K \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right) dx \wedge dy.$$

One of the classic applications of Theorem 3.1 is computing the area of a domain. For any compact domain $K \subset \mathbb{R}^2$, we have:

$$\int_{\partial K} xdy = \int\int_K dxdy, \text{ and } \int_{\partial K} ydx = -\int\int_K dxdy.$$

Thus, the area $A(K)$ of K can be computed using a line integral:

$$A(K) = \int\int_K dxdy = \int_{\partial K} xdy = -\int_{\partial K} ydx = \frac{1}{2}\int_{\partial K} xdy - ydx.$$

3.4.2 Green-Riemann Formula in $\mathbb{R}^3$

The Green-Riemann formula extends to differential forms of degree 1 and of class C^k defined on an open set $U \subset \mathbb{R}^3$. This formula is known as Stokes' Theorem in $\mathbb{R}^3$.

Theorem 3.2. Let K be a compact domain with a boundary of class C^1 (i.e., a curve of class C^1), and let $\phi : K \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be an injective parametrization of class C^1 defining an oriented surface $S = \phi(K)$ with boundary $\partial S = \phi(\partial K)$ (with an orientation compatible with that of S). Let $\omega = Pdx + Qdy + Rdz$ be a differential forms of degree 1 and of class C^1 on an open set $U \subset \mathbb{R}^3$. Then

$$\int_{\partial S} \omega = \int\int_S d\omega.$$

Recall that:

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Example 3.13. Let us compute the integral $\int_{\partial S} ydx + zdy + xdz$ both directly and using Stokes' Theorem. Consider the domain

$$K = \left\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\right\}$$

and the surface S in $\mathbb{R}^3$defined by the parametrization:

$$\begin{array}{rccl} \phi: & K & \to & S \subset \mathbb{R}^3 \\ & (u, v) & \mapsto & (x, y, z)(u, v) = (u, v, 1 - u^2 - v^2). \end{array}$$

The boundary ∂S is defined by the parametrization:

$$\begin{aligned} \psi: \ [0,2\pi] &\to \mathbb{R}^3 \\ t &\mapsto (x,y,z)\,(u,v) = (\cos t, \sin t, 0)\,. \end{aligned}$$

We calculate the line integral along ∂S:

$$\begin{aligned} \int_{\partial S} ydx + zdy + xdz &= \int_0^{2\pi} \left[\sin t\,(-\sin t) + 0\,(\cos t) + \cos t\,(0)\right] dt \\ &= -\int_0^{2\pi} \sin^2 t dt = -\pi. \end{aligned}$$

Now, using Stokes' Theorem, we calculate the surface integral

$$\begin{aligned} \int_{\partial S} ydx + zdy + xdz &= \int\int_S d\,(ydx + zdy + xdz) \\ &= \int\int_S -dx\wedge dy + dx\wedge dz - dy\wedge dz \end{aligned}$$

Pulling back the forms via ϕ, we obtain:

$$\int\int_S d\,(ydx + zdy + xdz) = \int\int_K (-1-2u-2v)\,du\wedge dv.$$

Switching to polar coordinates

$$u = r\cos\theta,\ v = r\sin\theta,\ 0\le r\le 1,\ 0\le\theta\le 2\pi,$$

the integral becomes

$$\int\int_K (-1-2u-2v)\,du\wedge dv = \int_0^1\int_0^{2\pi} (-1-2r\sin\theta-2r\cos\theta)\,rdrd\theta = -\pi.$$

This confirms the result obtained by the direct computation.

3.4.3 Ostrogradsky's Formula (Divergence Theorem)

Theorem 3.3. Let $\omega = Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy$ be a differential form of degree 2 and of class C^1 defined on an open set $U\subset\mathbb{R}^3$ containing a compact set K with a boundary of class C^1. If ∂K denotes the oriented boundary of K, then

$$\int\int_{\partial K} Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy = \int\int\int_K \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\wedge dy\wedge dz.$$

Remark 3.8. By setting $V = (P, Q, R)$, and $d\sigma = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$, Ostrogradsky's formula can be written more compactly as:

$$\int\int_{\partial K} V(x, y, z) .d\sigma = \int\int\int_{K} \operatorname{div} V(x, y, z)\, dxdydz.$$

This formula can be interpreted as: The integral of the divergence of the vector field V over the volume K equals the flux of V through the boundary ∂K.

3.5 Chapter Three Exercises

Exercise 3.1.

Let $U \subset \mathbb{R}^n$ be an open set, and let $\omega \in \Omega^p_{C^2}(U, \mathbb{R})$, $\alpha \in \Omega^r_{C^2}(U, \mathbb{R})$. Denote by d to the exterior derivative operator.

1. Prove that the exterior derivative is linear:

$$d(\omega + \alpha) = d(\omega) + d(\alpha).$$

2. Prove the Leibniz rule for the wedge product:

$$d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha.$$

3. Prove that d is nilpotent:

$$d(d\omega) = 0.$$

Exercise 3.2.

Let φ be the map from $\mathbb{R}$ to $\mathbb{R}^2$, and let ω be the differential form of degree 1 on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\varphi(t) = (\cos(t), \sin(t)) \quad \text{and} \quad \omega = \frac{1}{x^2 + y^2}(-ydx + xdy).$$

1. Calculate $\varphi^*\omega$ (the pullback of ω by φ).

2. Calculate the integral of ω over $\mathbb{S}^1$ (the unit circle).

3. Prove that ω is closed but not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$.

4. Explain why the Poincaré lemma does not apply to ω on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Exercise 3.3.

Let φ be the map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and let ω be the differential form of degree 2 on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\varphi(r,t) = (r\cos(t), r\sin(t)) \quad \text{and} \quad \omega = \frac{1}{x^2+y^2} dx \wedge dy.$$

1. Calculate $\varphi^*\omega$ (the pullback of ω by φ).

2. Let $D \subset \mathbb{R}^2$ be the annulus:

$$D = \left\{(x,y) \in \mathbb{R}^2 : r_1^2 \leq x^2+y^2 \leq r_2^2\right\} \text{ with } 0 < r_1 < r_2.$$

   Calculate $\int_D \omega$ for given r_1, r_2 and interpret the result geometrically.

3. Prove that ω is closed but not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Exercise 3.4.

Let $M \subset \mathbb{R}^3$ be the surface defined by the circular cone:

$$M = \left\{(x,y,z) \in \mathbb{R}^3 : \ z = \sqrt{x^2+y^2}, \ x^2+y^2 \leq r\right\}.$$

and let $f : M \longrightarrow \mathbb{R}$ be the function defined by:

$$f(x,y,z) = x^2+y^2.$$

1. Give a parametrization $\varphi(u_1,u_2)$ of the surface M in polar coordinates.

2. Form the matrix of the induced metric $G = (g_{ij}) = (\langle \partial_i\varphi, \partial_j\varphi \rangle)$, and calculate $\det G$.

3. Calculate the surface integral:

$$\int_M f(x,y,z)\, d\sigma = \int_D f(\varphi(u_1,u_2)) . \sqrt{\det G} du_1 du_2.$$

Exercise 3.5.

Let $M \subset \mathbb{R}^4$ be the graph of the function $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$, defined by:

$$h(x,y,z) = z$$

restricted to the domain:

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\right\}.$$

Define a parametrization:

$$\varphi(x, y, z) = (x, y, z, z)$$

Let $f : M \longrightarrow \mathbb{R}$ be the function defined by:

$$f(x, y, z, t) = t$$

1. Verify that $\varphi : D \longrightarrow M \subset \mathbb{R}^4$ is a regular parametrization of the submanifold M.

2. Form the matrix of the induced metric $G = (g_{ij}) = (\langle \partial_i \varphi, \partial_j \varphi \rangle)$, and calculate $\det G$.

3. Compute the integral of f over M using the formula:

$$\int_M f(x, y, z, t)\, d\sigma = \int_D f(\varphi(x, y, z)) . \sqrt{\det G} dx dy dz$$

Exercise 3.6.

Let γ be the oriented boundary of the domain bounded by the curves given by the equations $y = x^2$ and $x = y^2$.

1. Sketch the domain bounded by these two curves in the plane.

2. Verify that this domain is a compact region with a C^1 piecewise boundary.

3. Using Green–Riemann's formula, compute the line integral:

$$I = \int_\gamma \left(2xy - x^2\right) dx + \left(x + y^2\right) dy.$$

Exercise 3.7.

Let $a > 0$, $b > 0$, and consider the domain

$$D = \left\{(x, y) \in \mathbb{R}^2,\ x \geq 0,\ y \geq 0,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}.$$

1. Sketch the domain D.

2. Verify that this domain is a compact region with a C^1 piecewise boundary.

3. Using Green–Riemann's formula, calculate the double integral

$$J = \int\int_D \left(2x^3 - y\right) dxdy.$$

Exercise 3.8.

Let $a > 0$, and consider the parametric curve defined by:

$$x = a\left(t - \sin\left(t\right)\right), \quad y = a\left(1 - \cos\left(t\right)\right), \quad \text{for } t \in \left[0, 2\pi\right].$$

This curve traces a full arc of a cycloid, starting at the origin $(0, 0)$ and ending at $(2\pi a, 0)$. The curve lies entirely above the x-axis.

1. Sketch the domain enclosed between the cycloid and the x-axis.

2. Using Green's Formula, Prove the Area Formula

$$\text{Area} = \int_a^b x\left(t\right) y'\left(t\right) dt$$

3. Use the above formula to calculate the area enclosed between the cycloid and the x-axis.

Exercise 3.9.

Consider the surface

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^4,\ 0 \leq z \leq 1\right\},$$

and let $\Gamma = \partial\Sigma$ denote its positively oriented boundary. The parametrizations of Σ and Γ are given by::

$$\Sigma = \left\{\sigma\left(r, \theta\right) = \left(r^2\cos\left(\theta\right), r^2\sin\left(\theta\right), r\right),\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\right\},$$

$$\Gamma = \left\{\gamma\left(\theta\right) = \left(\cos\left(\theta\right), \sin\left(\theta\right), 1\right),\ 0 \leq \theta \leq 2\pi\right\}.$$

Consider the differential form ω of degree 1 on $\mathbb{R}^3$ defined by

$$\omega(x, y, z) = x^2 y dx + z dy + x dz.$$

1. Calculate the exterior derivative $d\omega$.
2. Calculate the surface integral $\int\int_{\Sigma} d\omega$.
3. Calculate the line integral $\int_{\Gamma} \omega$. Does Stokes'theorem holds?

Exercise 3.10.

Let $\omega \in \Omega^2(\mathbb{R}^3, \mathbb{R})$ be the differential form of degree 2 defined by

$$\omega(x, y, z) = xyz dx \wedge dy.$$

Let $V \subset \mathbb{R}^3$ be the unit ball defined by

$$V = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\right\}$$

and let $\Sigma = \partial V$ be the unit sphere, oriented with the outward normal.

1. Calculate the exterior derivative $d\omega$.
2. Evaluate the volume integral $\int\int\int_{V} d\omega$.
3. Evaluate the surface integral$\int\int_{\Sigma} \omega$.
4. Does the divergence theorem holds.

3.6 Solutions to Chapter Three Exercises

Exercise 3.2.

1. We have

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = f(x,y)\,dx + g(x,y)\,dy.$$

Then

$$\begin{aligned}\varphi^*\omega &= f(\cos(t),\sin(t))\,d(\cos(t)) + g(\cos(t),\sin(t))\,d(\sin(t)) \\ &= \frac{\sin^2(t)}{\cos^2(t)+\sin^2(t)}dt + \frac{\cos^2(t)}{\cos^2(t)+\sin^2(t)}dt \\ &= dt \\ &= h(t)\,dt.\end{aligned}$$

2. Integral of ω over $\mathbb{S}^1$, we know that $\mathbb{S}^1 = \{(\cos(t),\sin(t)),\ 0 < t \leq 2\pi\}$. Then

$$\int_\varphi \omega = \int_0^{2\pi} h(t)\,dt = 2\pi.$$

3. Prove that ω is closed but not exact on $\mathbb{R}^2\backslash\{(0,0)\}$.

(a) ω is closed: Write $\omega = Pdx + Qdy$, where

$$P = -\frac{y}{x^2+y^2},\ Q = \frac{x}{x^2+y^2}.$$

Compute the exterior derivative:

$$\begin{aligned}d\omega &= d(Pdx+Qdy) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy \\ &= 0.\end{aligned}$$

(b) ω is not exact. Suppose ω were exact. Then there would exist a function $f : \mathbb{R}^2\backslash\{(0,0)\} \longrightarrow \mathbb{R}$ such that $\omega = df$. In that case, for any closed curve γ (i.e., a curve with the same start and end point), the integral of ω over γ would be zero:

$$\int_\gamma \omega = \int_\gamma df = 0.$$

However, from part 2, for the closed curve $\mathbb{S}^1$(the unit circle), we have:

$$\int_{\mathbb{S}^1} \omega = 2\pi \neq 0.$$

This contradiction shows that ω cannot be exact.

4. The Poincaré lemma states that every closed form is exact on a star-shaped domain. However, $\mathbb{R}^2 \backslash \{(0,0)\}$ is not star-shaped (it has a hole at the origin). Therefore, the lemma does not apply, and closed forms need not be exact.

Exercise 3.3.

1. We have

$$\varphi(r,t) = (r\cos(t), r\sin(t)) \quad \text{and} \quad \omega = \frac{1}{x^2+y^2} dx \wedge dy.$$

As in the Exercise 3.2, we find

$$\varphi^* \omega = \frac{1}{r} dr \wedge dt.$$

2. Integral of ω over D. Using the pullback from question 1, we have

$$\int_D \omega = \int_{r_1}^{r_2} \int_0^{2\pi} \frac{1}{r} dr \wedge dt = 2\pi \ln\left(\frac{r_2}{r_1}\right).$$

Exercise 3.4.

1. Parametrization $\varphi(\rho,\theta)$**.**The surface is the cone $z = \sqrt{x^2+y^2}$, with the radial restriction $x^2+y^2 \leq r$. Put $\rho = \sqrt{x^2+y^2}$. Then $0 \leq \rho \leq \sqrt{r}$ and $\theta \in [0, 2\pi[$. Using polar coordinates in the xy-plane:

$$\varphi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho), \quad 0 \leq \rho \leq \sqrt{r},\ 0 \leq \theta < 2\pi.$$

2. Induced metric $G = (g_{ij})$ and $\det G$. We have

$$\frac{\partial\varphi}{\partial\rho} = (\cos\theta, \sin\theta, 1), \ \frac{\partial\varphi}{\partial\theta} = (-\rho\sin\theta, \rho\cos\theta, 0),$$

and

$$g_{11} = \left\langle \frac{\partial \varphi}{\partial \rho}, \frac{\partial \varphi}{\partial \rho} \right\rangle = \cos^2\theta + \sin^2\theta + 1 = 2,$$
$$g_{12} = g_{21} = \left\langle \frac{\partial \varphi}{\partial \rho}, \frac{\partial \varphi}{\partial \theta} \right\rangle = -\rho\cos\theta\sin\theta + \rho\sin\theta\cos\theta = 0,$$
$$g_{22} = \left\langle \frac{\partial \varphi}{\partial \theta}, \frac{\partial \varphi}{\partial \theta} \right\rangle = \rho^2\sin^2\theta + \rho^2\cos^2\theta = \rho^2.$$

Thus

$$G = \begin{pmatrix} 2 & 0 \\ 0 & \rho^2 \end{pmatrix}, \quad \det G = 2\rho^2.$$

3. Surface integral $\int_M f(x,y,z)\,d\sigma$. Here $f(x,y,z) = x^2+y^2 = \rho^2$ on the parametrization, and $\sqrt{\det G} = \sqrt{2}\rho$. Therefore

$$\int_M f d\sigma = \int_0^{2\pi}\int_0^{\sqrt{r}} \rho^2\sqrt{2}\rho d\rho d\theta = \sqrt{2}\int_0^{2\pi} d\theta \int_0^{\sqrt{r}} \rho^3 d\rho$$

So the value of the surface integral is

$$\int_M f d\sigma = \frac{\pi r^2}{\sqrt{2}}$$

Exercise 3.5.

1. To show that φ is a regular parametrization, we need to check that:

   - φ is smooth (which it is, as each component is linear).
   - φ is injective: if $\varphi(x_1,y_1,z_1) = \varphi(x_2,y_2,z_2)$, then $(x_1,y_1,z_1) = (x_2,y_2,z_2)$.
   - The differential $d\varphi$ has full rank (i.e., rank 3) at every point a in D. Compute the Jacobian matrix of φ :

$$J_\varphi(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix has rank 3 (the first three rows are linearly independent). Thus, φ is an immersion. Also, φ is clearly injective. Therefore, φ is a regular parametrization.

2. Induced metric $G = (g_{ij})$ and $\det G$. The induced metric is given by:

$$g_{ij} = \left\langle \frac{\partial \varphi}{\partial u^i}, \frac{\partial \varphi}{\partial u^j} \right\rangle, \text{ where } \left(u^1, u^2, u^3\right) = (x, y, z).$$

Compute the partial derivatives:

$$\frac{\partial \varphi}{\partial x} = (1, 0, 0, 0), \ \frac{\partial \varphi}{\partial y} = (0, 1, 0, 0), \ \frac{\partial \varphi}{\partial z} = (0, 0, 1, 1).$$

Now compute the inner products (using the standard Euclidean inner product in $\mathbb{R}^4$):

$$g_{11} = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial x} \right\rangle = 1, \ g_{12} = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right\rangle = 0, \ g_{13} = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial z} \right\rangle = 0,$$
$$g_{22} = \left\langle \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial y} \right\rangle = 1, \ g_{23} = \left\langle \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle = 0, \ g_{33} = \left\langle \frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial z} \right\rangle = 2.$$

So the metric matrix $G = (g_{ij})$ and $\det G$ are:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \det G = 2.$$

3. Compute the integral of f over M. We have

$$\int_M f(x, y, z, t)\, d\sigma = \int_D f(\varphi(x, y, z)) \sqrt{\det G} dx dy dz.$$

Here, $f(x, y, z, t) = t$, and on M we have $t = z$ (since $\varphi(x, y, z) = (x, y, z, z)$). So:

$$f(\varphi(x, y, z)) = z,$$

and

$$\begin{aligned} \int_M f(x, y, z, t)\, d\sigma &= \int_D z\sqrt{2} dx dy dz \\ &= \sqrt{2} \int_{x^2+y^2 \le 1} \int_{z=0}^{1} z dx dy dz. \end{aligned}$$

So

$$\int_M f d\sigma = \frac{\pi\sqrt{2}}{2}.$$

Exercise 3.6.

1. Compactness and boundary regularity. The region D is closed (as it includes its boundary) and bounded (since it is contained in, say, $[0,1] \times [0,1]$, so it is compact.

   The boundary consists of two curves: $\gamma_1 : y = x^2$ for $x \in [0,1]$ and $\gamma_2 : y = \sqrt{x}$ for $x \in [0,1]$. Both are of class C^1 on $]0,1[$. However, At $x = 0$,

   - The derivative of $\sqrt{x}$ is undefined (infinite derivative), so there is a vertical tangent.
   - The derivative of x^2 is 0

   At $x = 1$

   - The derivative of $\sqrt{x}$ is $1/2$
   - The derivative of x^2 is 2

     So, the curves intersect at distinct angles at both $(0,0)$ and $(1,1)$, resulting in corners at these points. Thus, the boundary is piecewise C^1 (each smooth piece is C^1). This level of regularity is sufficient for the application of Green's Theorem, which requires the boundary to be piecewise smooth.

2. Consider the domain

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ x^2 \leq y \leq \sqrt{x}\right\}.$$

   Let us define the functions $P(x,y) = 2xy - x^2$ and $Q(x,y) = x + y^2$. We aim to evaluate the integral

$$I = \int\int_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

   First, compute the integrand:

$$\frac{\partial Q}{\partial x} = 1,\ \frac{\partial P}{\partial y} = 2x.$$

Thus, the integral becomes

$$\begin{aligned} I &= \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (1-2x)\, dy \right) dx \\ &= \int_0^1 (1-2x) \left(\sqrt{x} - x^2 \right) dx \\ &= \int_0^1 \left(\sqrt{x} - 2x\sqrt{x} - x^2 + 2x^3 \right) dx \\ &= \frac{1}{30}. \end{aligned}$$

Exercise 3.7.

1. We will use Green-Riemann's formula. To do this, we start by finding P and Q such that

$$\frac{\partial Q}{\partial x}(x,y) = 2x^3 \text{ and } \frac{\partial P}{\partial y}(x,y) = y.$$

For example, we take $Q(x,y) = \frac{1}{2}x^4$ and $P(x,y) = \frac{1}{2}y^2$. The domain considered is the interior of a quarter ellipse, which can be parameterized by

$$x = a\cos(\theta),\ y = b\sin(\theta),\ \theta \in \left[0, \frac{\pi}{2}\right].$$

We then have

$$\begin{aligned} I &= \int_{\partial D} P(x,y)\, dx + Q(x,y)\, dy \\ &= \int_0^{\frac{\pi}{2}} \left[-\frac{b^2 \sin^2(\theta)}{2} (-a\sin(\theta)) + \frac{a^4 \cos^4(\theta)}{2} (b\cos(\theta)) \right] d\theta. \end{aligned}$$

By evaluating this integral, we find

$$I = \frac{4}{15}a^4 b - \frac{ab^2}{3}.$$

Exercise 3.8.

1. Let γ be the boundary of the compact domain. Then, by Green–Riemann's theorem (up to a sign), we know that:

$$A = \int_\gamma x dy = -\int_\gamma y dx = \frac{1}{2}\int_\gamma x dy - y dx.$$

Let us use the second expression. Note that the integral of ydx along the $y-$axis vanishes, since we can parametrize that portion by; we can use the parametrization $x(t) = 0$, $y(t) = t$, which yields $dx = 0$. Therefore, we focus on the remaining part of the curve:

$$\begin{aligned} A &= \int_0^{2\pi} a(1-\cos(t)) \times a(1-\cos(t))\, dt \\ &= a^2 \int_0^{2\pi} \left(1 - 2\cos(t) + \cos^2(t)\right) dt \\ &= a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos(t) + \frac{\cos(2t)}{2}\right) dt \\ &= 3\pi a^2 \text{ units of area} \end{aligned}$$

Exercise 3.9.

1. We have:

$$d\omega(x,y,z) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

$$d\omega(x,y,z) = -dy \wedge dz - x^2 dx \wedge dy.$$

2. We compute:

$$\begin{aligned} \int\int_\Sigma d\omega &= \int_0^1 \int_0^{2\pi} \begin{bmatrix} -(2r\sin(\theta)\, dr + r^2\cos(\theta)\, d\theta) \wedge dr \\ -r^4\cos^2(\theta)\,(2r\cos(\theta)\, dr - r^2\sin(\theta)\, d\theta) \wedge (2r\sin(\theta)\, dr + r^2\cos(\theta)\, d\theta) \end{bmatrix} \\ &= \int_0^1 \int_0^{2\pi} \left[r^2\cos(\theta) - 2r^7\cos^2(\theta)\right] dr d\theta = -\frac{\pi}{4}. \end{aligned}$$

3. We compute:

$$\begin{aligned} \int_\Gamma \omega.d\Gamma &= \int_0^{2\pi} \omega(\gamma(\theta)).\gamma'(\theta)\, d\theta \\ &= \int_0^{2\pi} \left(\cos^2(\theta)\sin(\theta), 1, \cos(\theta)\right).(-\sin(\theta), \cos(\theta), 0)\, d\theta \\ &= -\int_0^{2\pi} \cos^2(\theta)\sin^2(\theta)\, d\theta = -\frac{\pi}{4}. \end{aligned}$$

4. Since $\int\int_\Sigma d\omega = \int_\Gamma \omega.d\Gamma$, Stokes' theorem is verified.

Exercise 3.10.

1. The parametrization

$$(r, \theta, \varphi) \mapsto (r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi))$$

with $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$, and $0 \leq r \leq 1$. The Jacobian is $-r^2\sin(\varphi)$, and since $d\omega = xydx \wedge dy \wedge dz$, we have:

$$\begin{aligned}\int\int\int_V d\omega &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} -r^4\cos(\theta)\sin(\theta)\sin^3(\varphi)\, dr d\theta d\varphi \\ &= 0.\end{aligned}$$

2. The surface Σ is parametrized by

$$\eta : (\theta, \varphi) \mapsto (\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi)),$$

with $0 \leq \theta \leq 2\pi$, and $0 \leq \varphi \leq \pi$. We have

$$\omega = xyzdx \wedge dy = \cos(\theta)\sin^3(\varphi)\sin(\theta)\cos^2(\varphi)\, d\varphi \wedge d\theta$$

and

$$\int\int_\Sigma \omega = \int_0^{2\pi} \int_0^{\pi} \cos(\theta)\sin^3(\varphi)\sin(\theta)\cos^2(\varphi)\, d\varphi d\theta = 0.$$

3. This is consistent with the divergence Theorem: $\int\int\int_V d\omega = \int\int_\Sigma \omega$.

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